

# Structural analysis for fault diagnosis of models with constraints

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**Abstract:** Structural analysis is a standard tool to identify submodels that can be used to design model based diagnostic tests. Structural approaches typically operate on models described by a set of equations. This work extends such methods to be able to handle models with constraints, e.g. inequality constraints on state variables. The objective is to improve isolability properties of a diagnosis system by extending the class of redundancy relations. An algorithm is developed that identifies which are the constraints and equations that can be used together to derive a new test that can not be found using previous approaches.

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## 1. INTRODUCTION

The task of detecting faults and isolating the faulty component is an important topic in many applications. In this paper, passive model based diagnosis systems are considered, i.e. the diagnosis system passively uses the observations to make a diagnostic decision based on a formal model of the process under supervision. A common architecture for such fault diagnosis systems is to have a set of pre-compiled tests, or fault detectors, and then use the results from the set of detectors in a fault isolation unit to perform the isolation.

A main difficulty in the development of a diagnostic system is to design the diagnostic tests. An approach that has proven beneficial is to analyze the structure of the model to identify, possibly small, submodels that can be used to design tests. A test, based on the submodel, then checks if the observations are consistent with the submodel. For this task, there exists many proposed solutions, e.g. matching procedures, observer designs, numerical solvers, and elimination approaches. One advantage with using submodels for the design of detectors is that only parts of the model, not the complete model, need to be considered when designing each test which may make the design simpler. Also, fault isolation is made possible by considering different and well selected submodels when designing each test. Such submodels are called Minimal Structurally Overdetermined sets of equations (MSO) Krysander et al. [2008], or Analytical Redundancy Relations (ARR) Blanke et al. [2003], or Minimal Evaluation Chain (MEC) Pulido and Gonzalez [2004]. Here, the term MSO will be used.

The referred approaches above handle models that consist of a set of relations, typically equations, and produce a scalar residual where an alarm is generated if the residual deviates significantly from zero. The main contribution of this work is the extension of such approaches to handle also cases where the model includes inequalities or more general relations. This has the effect that the resulting test may be a set membership expression.

The outline of the paper is as follows. Section 2 gives some motivating examples and the proposed algorithm, based on the structure of the model, is outlined in Section 3. To formally motivate the approach, the solution is analytically characterized in Section 4 and then the analytical characterization is translated into a structural characterization in Section 5. The key step in the algorithm

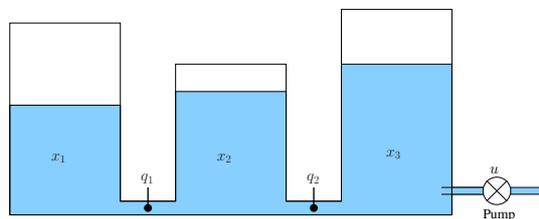


Fig. 1. Coupled water tank system.

outlined in Section 3 is described in detail in Section 6. The method is illustrated using a dynamic model of coupled water tanks in Section 7.

## 2. MOTIVATING EXAMPLES

This section discusses the objectives of the paper using two motivating examples. The first example focuses on the problem formulation while the second is a smaller example used to illustrate the theory used in the approach.

### 2.1 First example

The first example is the system with three coupled tanks shown in Figure 1. Assume that available measurements are the level  $x_1$  in tank 1 and the flow  $q_1$  between tank 1 and 2. The pump power  $u$  is also known. A normalized first principles model of the system and observations is then given by the following set of equations:

$$\begin{aligned} e_1 : \dot{x}_1 &= -q_1 \\ e_2 : \dot{x}_2 &= q_1 - q_2 \\ e_3 : \dot{x}_3 &= q_2 - u \\ e_4 : q_1 &= x_1 - x_2 \\ e_5 : q_2 &= x_2 - x_3 \\ e_6 : y_1 &= x_1 \\ e_7 : y_2 &= q_1 \end{aligned}$$

where  $x_i$  are tank levels,  $q_i$  flows,  $y_i$  measurements and  $u$  known control input. As mentioned in the introduction, one approach to design tests for a given model is to first analyze the structure of the model to identify submodels that can be used for test design. The structure of the tank system model is given in Figure 2.

The structural model shows which unknowns signals are included in each equation. A minimal structurally overdetermined (MSO) set of equations is a set with the property

equation $E$	unknown $X$				
	$x_1$	$x_2$	$x_3$	$q_1$	$q_2$
$e_1$	X			X	
$e_2$		X		X	X
$e_3$			X		X
$e_4$	X	X		X	
$e_5$		X	X		X
$e_6$	X				
$e_7$				X	

Fig. 2. Structural model of the tank system.

that there are more equations than unknown signals and that no proper subset has the same property. Such sets can be used to define diagnostic tests. By analyzing the structural model, all MSO sets can be obtained and can thus be used to define a set of tests Krysander et al. [2008]. One way of transferring an MSO into a test is to eliminate all unknown signals in the MSO set which results in a single equation involving known signals only, which can be used to check consistency of the submodel. In the water tank system the MSO sets and their corresponding redundancy relations are:

$$\begin{aligned} \{e_1, e_6, e_7\} : \dot{y}_1 + y_2 &= 0 \\ \{e_2, e_3, e_4, e_5, e_6, e_7\} : \dot{y}_1 + 2\dot{y}_1 - \dot{y}_2 - 3\dot{y}_2 - y_2 &= 0 \\ \{e_1, e_2, e_3, e_4, e_5, e_6\} : y_1^{(3)} + 4\ddot{y}_1 + 3\dot{y}_1 + u &= 0 \\ \{e_1, e_2, e_3, e_4, e_5, e_7\} : \dot{y}_2 + 3\dot{y}_2 + 3y_2 - u &= 0 \end{aligned}$$

These 4 tests give a certain isolability performance of the diagnostic system. However, since the tanks are sealed we know that the following constraints are always valid

$$\begin{aligned} c_1 : x_1 &\in [0, h_1] \\ c_2 : x_2 &\in [0, h_2] \\ c_3 : x_3 &\in [0, h_3] \end{aligned}$$

where  $h_i$  is the height of each tank. The main contribution of this paper is to extend the structural analysis to also include the inequality constraints in the diagnostic analysis. This will make it possible to systematically find more submodels that can be used for test construction and therefore possibly increase the isolability performance of the diagnostic system.

Now, using the model together with the constraints it is possible to create new tests. For example, using the set  $\{e_6, c_1\}$  we immediately obtain the redundancy relation  $y_1 \in [0, h_1]$ . Another redundancy relation is  $y_2 = x_1 - x_2 \in [-h_2, h_1]$  where the set  $\{e_4, e_7, c_1, c_2\}$  was used to create the test. The physical interpretation of this test is that the largest flow in each direction occur when one tank is full and the other is empty. Another example with similar physical interpretation is

$$\dot{y}_1 = \dot{x}_1 = -q_1 = x_2 - x_1 \in [-h_1, h_2] \quad (1)$$

where the set  $\{e_1, e_4, e_6, c_1, c_2\}$  was used. This example illustrates how differentiation of equations, in this case of equation  $e_6$ , can be combined with algebraic methods to design tests. The example above also illustrates how constraints can be included both in the system model and in the test. We will return to this example in Section 7.

## 2.2 Second example

The first example illustrates what type of problems the suggested approach can handle. A second example will now be discussed that consists of a small, static model where the main objective is to illustrate principles that can be used in the development of the algorithm in Sections 4 and 5. Thus, consider the model

$$x_1^2 + x_2^2 = 1 \quad (2)$$

$$y_1 = x_1 \quad (3)$$

$$y_2 = x_1 + x_2 \quad (4)$$

where  $y_1$  and  $y_2$  are known measurements and  $x_i$  are unknown variables. This set of equations is an MSO set with three equations and two unknowns. It is therefore possible to design a test by eliminating the unknowns to obtain a single equation

$$y_1^2 + (y_2 - y_1)^2 = 1 \quad (5)$$

which only includes known signals and can directly be used to form a test

$$|y_1^2 + (y_2 - y_1)^2 - 1| > J \quad (6)$$

for some positive threshold  $J$ .

For a system of linear equations it is necessary that the set of equations is overdetermined to be able to define a test. However, in the example above the equation (2) is a non-linear constraint and this fact opens up possibilities to use smaller sets of equations to construct tests. For example, (2) and (4) imply that

$$y_2^2 = 2 - (x_1 - x_2)^2 \leq 2 \quad (7)$$

and based on this inequality, the following test can be constructed

$$y_2^2 - 2 > J \quad (8)$$

It is easy to see that (8) implies (6) while the converse implication is not valid. Hence, test (8) has weaker detectability properties than (6) and for detectability purposes it is no gain in introducing (8). But, as we will show, (8) contributes to fault isolation performance. To see this, consider faults in the two sensors. Test (6) is sensitive to both these faults but cannot isolate one faulty sensor. Test (8) on the other hand is only sensitive to fault in sensor  $y_2$  which makes it possible to isolate fault in sensor  $y_2$  from fault in sensor  $y_1$ .

A test consists of an expression involving only known variables which means that elimination of unknown variables is a key step when deriving a test. To derive tests, consisting of an equality constraint like in (5), classical elimination theory can be used. Elimination is direct for linear systems and for non-linear systems Gröbner basis theory Cox et al. [1991] is a standard tool for performing elimination. To incorporate tests in the form (8) in the framework, the class of redundancy relations has to be extended to include inequalities as well. Considering this extended class of redundancy relations, one method to perform elimination is quantifier elimination theory Arnon [1988], Jirstrand [1997].

## 2.3 Problem formulation

To summarize the discussion above we conclude that it is beneficial for fault isolation purposes to include redundancy relations formulated as inequalities. Using small subsets of model equations is a standard tool Krysander et al. [2008], Blanke et al. [2003], Pulido and Gonzalez [2004] to obtain equality relations. These works do not cover models with inequalities or the case with inequality redundancy relations that allows for even smaller subset of model equations to be used for deriving tests. The main objective, which is formally treated in the following sections, is to find all subsets of model equations that can be used to derive tests using this extended framework.

## 3. APPROACH OUTLINE

In the previous section we found how inequality relations can be used to construct a test that increases the isolability

performance. This section will formalize a procedure how to derive such relations in a systematic way. First static models are considered and the model (2)–(4) will be used to illustrate the proposed procedure. The section is concluded with a general algorithm.

In the approach, the model is partitioned into a set of linear equations  $E$  and a set of non-linear constraints and inequalities. The objective is then to determine which subsets of the linear equations  $E$  and the non-linear constraints that can be used to design a test. It is assumed that tests derived from submodels with only linear constraints are found with some of the standard methods described in the introduction. These tests are not considered in this work.

Now, the example in Section 2.2 will be used to illustrate how all useful tests can be found in a systematic way. To proceed, let  $\mathcal{M}_0$  denote the set of consistent values of the unknown variables in the non-linear constraint, i.e.

$$\mathcal{M}_0 = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\} \quad (9)$$

In general, the set is parametrized by the known signals denoted by  $z$  if these are included in the non-linear relations. Given a set  $\mathcal{M}_0$ , the next step is to find subsets  $E'$  of the set of linear equations

$$E_0 = \{y_1 = x_1, y_2 = x_1 + x_2\} \quad (10)$$

such that  $\mathcal{M}_0$  and  $E'$  can be used to construct a test. In the second motivating example, two sets of linear equations were used,  $E' = E_0$  and  $E' = \{y_2 = x_1 + x_2\}$  to construct tests (6) and (8).

The first subset,  $\{y_1 = x_1, y_2 = x_1 + x_2\}$ , defines a point in  $\mathbb{R}^2$  and the second,  $\{y_2 = x_1 + x_2\}$ , defines a line. In general, a subset of linear equations defines an affine subspace, depending on the known variables  $z$ . The geometrical interpretation of the test is to decide whether the affine subspace intersects the set  $\mathcal{M}_0$  or not.

In addition to the two tests above, one more useful test can be derived from the model in the example. By eliminating the  $x_2$ -variable in  $\mathcal{M}_0$ , the set

$$\mathcal{M}_0^1 = \{(x_1) : \exists x_2 (x_1, x_2) \in \mathcal{M}_0\} = \{(x_1) : x_1^2 \leq 1\}$$

is obtained. Since  $\mathcal{M}_0^1$  is a proper subset of  $\mathbb{R}$ , it may be used to design tests. Now  $\mathcal{M}_0^1$  is used in the same way as  $\mathcal{M}_0$  and the set  $E' = \{y_1 = x_1\}$  can be used together with  $\mathcal{M}_0^1$  to obtain the relation

$$y_1^2 \leq 1 \quad (11)$$

which also can be used as a test. If, instead of  $x_2$ , the  $x_1$ -variable is eliminated, the set

$$\mathcal{M}_0^2 = \{(x_2) : \exists x_1 (x_1, x_2) \in \mathcal{M}_0\} = \{(x_2) : x_2^2 \leq 1\}$$

is obtained. The only set  $E'$  that together with  $\mathcal{M}_0^2$  provides redundancy is  $E' = \{y_1 = x_1, y_2 = x_1 + x_2\}$ . The corresponding test is  $(y_2 - y_1)^2 \leq 1$ .

This test is sensitive to the same faults as redundancy relation (5), but has weaker detectability properties, and does therefore not contribute neither to fault detectability nor to fault isolability. This test should therefore be omitted and a characterizing property, which can be used to efficiently exclude this test, is that the variable  $x_2$  was eliminated in both the linear equations  $E'$  and the original constraint  $\mathcal{M}_0$ . Line 9 in the algorithm below performs this operation.

It has been shown how all tests can be found in a systematic way for the small example. Now we will show how to formulate a procedure for a general model with a set of constraints  $\mathcal{M}_i$ ,  $i \in I$ , and a set of linear equations  $E_0$ .

The tests in the examples were derived by considering different subsets of constraints and variables. The procedure is generalized as follows. First a subset of the constraints  $\mathcal{M}_i$ ,  $i \in I' \subset I$  is chosen, which defines a constraint  $\mathcal{M}'$  with variables  $\mathcal{X}'$ . For example, in Section 2.1 the set  $\mathcal{M}' = \{(x_1, x_2) : x_1 \in [0, h_1], x_2 \in [0, h_2]\}$  was used to derive the test  $y_2 \in [-h_2, h_1]$ . For each such constraint  $\mathcal{M}'$ , it is possible to derive tests by considering different subsets  $\mathcal{X}'_1$  of the variables  $\mathcal{X}'$ . For example test (11) was derived considering the set  $\mathcal{X}'_1 = \{x_1\}$ . For each combination of subsets of constraints and variables, a constraint is obtained and denoted by  $\mathcal{M}$ . For each such constraint, we then compute the set of all subsets of the linear equations  $E$  that can be used to create a test. This set will be denoted by M-TestSet( $\mathcal{M}, E$ ) and the corresponding algorithm is defined in Section 6. The algorithm is outlined in the pseudo-code below and the algorithm output TS is used together with analytical elimination tools to derive tests.

```

1  function TS=TestSet( $\{\mathcal{M}_i\}_{i \in I}, E_0$ )
2  TS =  $\emptyset$ 
3  for each  $I' \subseteq I$ 
4     $\mathcal{X}' = \cup_{i \in I'} \text{var}(\mathcal{M}_i)$ 
5     $\mathcal{M}' = \cap_{i \in I'} \mathcal{M}_i$ 
6    for each  $\mathcal{X}'_1 \subseteq \mathcal{X}'$ 
7       $\mathcal{X}_e = \mathcal{X}' \setminus \mathcal{X}'_1$ 
8       $\mathcal{M} = \{X_1 : \exists X_e. (X_1, X_e) \in \mathcal{M}'(z)\}$ 
9       $E = \{e \in E_0 : \text{var}(e) \cap \mathcal{X}_e = \emptyset\}$ 
10     TS = TS
11      $\cup \{ \langle \mathcal{M}, E' \rangle : E' \subseteq E \wedge E' \in \text{M-TestSet}(\mathcal{M}, E) \}$ 
12   end
13 end
14 TS=Remove_non-minimal(TS);

```

The set  $\text{var}(\mathcal{M}_i)$  is the set of variables that  $\mathcal{M}_i$  is defined on. The intersection operation in line 5 should be interpreted as the set of all  $X'$  such that the subset  $\text{var}(\mathcal{M}_i) \subset \mathcal{X}'$  belong to  $\mathcal{M}_i$  for all  $i \in I'$ . The minimality condition used in the last line is defined in Section 4.2.

In the sections that follow, the approach will be developed. First an analytical specification of the elements in the set M-TestSet( $\mathcal{M}, E$ ) is presented. Since the objective is to develop an algorithm using structural models, the analytical specification is translated into a set of structural conditions. Using these conditions, an algorithm is given in Section 6.

#### 4. ANALYTICAL CHARACTERIZATION

To analytically characterize the set M-TestSet( $\mathcal{M}, E$ ) we need to introduce some notations. Let  $z$  denote a vector that contains all known variables and let the constraint be defined by  $(x_{11}, \dots, x_{1n}) \in \mathcal{M}(z)$  where  $\mathcal{M}(z)$  is a subset of  $\mathbb{R}^n$  for every  $z$ . The set of linear equations is denoted by  $E = \{e_1, \dots, e_m\}$  and can be written in the form

$$E : A(z)X = b(z)$$

The set of unknowns in the linear equations is partitioned as  $X \in \mathbb{R}^N$ ,  $N \geq n$ ,  $X = (X_1, X_2)$  where  $X_1 = (x_{11}, \dots, x_{1n})$  is the set of variables used to define the constraint above, and  $X_2 = (x_{21}, \dots, x_{2m})$  are the remaining variables.

##### 4.1 Analytical conditions

Given a constraint set  $\mathcal{M}$  and the set of linear equations  $E$ , the task is now to find a subset  $E' : A'(z)X = b'(z)$  of the

linear equations, that together with the constraint set can be used to construct a test. The first step in the analysis is to specify which analytical properties the sought set  $E'$  must have.

Since it was assumed in Section 3 that all tests derived using only the linear equations are treated separately, a first condition is that the set of linear equations

$$E' : A'(z)X = b'(z)$$

is consistent for any right-hand side  $b(z)$ , i.e. matrix  $A'(z)$  has full row rank.

Solutions to the linear equations  $E'$  can be used to define the affine subspace of  $\mathbb{R}^n$ :

$$\mathcal{L} = \{X_1 : \exists X_2, A'(z)X = b'(z)\} \quad (12)$$

To test inconsistency between  $E'$  and  $\mathcal{M}$  is then equivalent to check if  $\mathcal{L} \cap \mathcal{M}$  is empty. A necessary requirement on  $E'$  is then that  $\mathcal{L}$  is a proper subset of  $\mathbb{R}^n$ , i.e.  $\dim \mathcal{L} < n$ .

If a proper subset of  $E'$  defines the same affine subspace  $\mathcal{L}$  as  $E'$ , then  $E'$  does not contribute to the performance and should be omitted. Thus, a condition is that no proper subset of  $E'$  defines the same affine subspace  $\mathcal{L}$ .

The last condition is that the set  $\mathcal{L}$  can not be represented as

$$\mathcal{L} = \{(x_{11}, \dots, x_{1n}) : (x_{11}, \dots, x_{1(j-1)}, x_{1(j+1)}, \dots, x_{1n}) \in \mathcal{L}^*, x_j \in \mathbb{R}\} \quad (13)$$

for some  $j$  and affine subspace  $\mathcal{L}^* \subset \mathbb{R}^{n-1}$ . If a set in this form can be used to construct a test, then it will be found when the set of variables  $\mathcal{X}_1$ , in the for-loop starting on line 6 in the algorithm, is some subset of the variables  $\{x_{11}, \dots, x_{1(j-1)}, x_{1(j+1)}, \dots, x_{1n}\}$ . All sets in this form are therefore excluded to avoid that the same test is found more than once. To illustrate this, using the example above, consider the case where  $\mathcal{X}_1 = \{x_1, x_2\}$  and  $\mathcal{M} = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ . Then the affine space  $\mathcal{L} = \{(x_1, x_2) : y_1 = x_1\}$  can be used to construct the test  $y_1^2 \leq 1$ . In this case,  $\mathcal{L}$  can be written in the form (13) with  $\mathcal{L}^* = \{(x_1) : y_1 = x_1\}$  and the same test can be obtained using the set of variables  $\mathcal{X}_1 = \{x_1\}$  and the sets  $\mathcal{M} = \{(x_1) : x_1^2 \leq 1\}$  and  $\mathcal{L} = \{(x_1) : y_1 = x_1\}$ .

To summarize the discussion above, the set  $E' : A'(z)X = b'(z)$  should fulfill the following conditions:

- The matrix  $A'(z)$  has full row rank.
- The set  $\mathcal{L}$  is an affine subspace of  $\mathbb{R}^n$  of dimension less than  $n$ .
- No proper subset of  $E'$  defines the same affine subspace  $\mathcal{L}$ .
- The affine subspace  $\mathcal{L}$ , defined by  $E'$ , can not be represented as in (13).

The conditions above will be used to exclude sets of linear equations that either can not be used to form a test or that give a test found in another step in the algorithm. However, the conditions do not guarantee that the set will give a test. For example, the set  $\mathcal{M} = \{(x_1, x_2) : x_1 = x_2^3\}$  has a non-empty intersection with any line in the plane and hence  $\mathcal{L}$  can not be one-dimensional. Sets of this type will be excluded in the final stage of the approach where the analytical elimination is performed.

#### 4.2 Minimality

In the example in Section 2.1, the set  $S = \{e_4, e_6, e_7, c_1, c_2\}$  fulfills the constraints a)-d) above and gives to the test

$$(y_1, y_2) \notin \{(y_1, y_2) : y_1 \in [0, h_1], y_1 - y_2 \in [0, h_2]\}$$

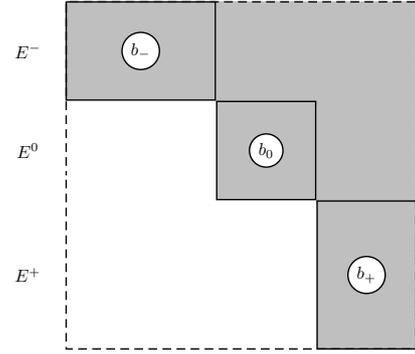


Fig. 3. The Dulmage-Mendelsohn decomposition.

The condition  $y_1 \in [0, h_1]$  can be derived from the set  $S_1 = \{e_6, c_1\}$  and  $y_1 - y_2 \in [0, h_2]$  from  $S_2 = \{e_4, e_6, e_7, c_2\}$ . The sets  $S_1$  and  $S_2$  also satisfy the constraints a)-d) and will thus be found by the algorithm. Further, the set  $S$  does not provide any additional detectability or isolability properties and should therefore be omitted. We will therefore exclude sets where a proper subset also fulfills conditions a)-d) and this is done by the function `Remove_non-minimal` used in function `TestSet` in Section 3.

## 5. STRUCTURAL CHARACTERIZATION

The objective of this section is to translate the analytical characterization from the previous section into a corresponding structural characterization. First, basic tools and notations from structural analysis are presented and then the structural characterization is given.

### 5.1 Theoretical background

Redundancy is a key property for models used in diagnosis. For a set of equations  $E$ , the structural redundancy is defined as

$$\varphi(E) = |E| - |\text{var}(E)|$$

where  $\text{var}(E)$  is the set of unknown variables in the set of equations. A set of equations with positive redundancy has more equations than unknown variables, and is therefore called structurally overdetermined. By permuting rows and columns of the incidence matrix of the model one can obtain the Dulmage-Mendelsohn decomposition which partitions the set of model equations into three parts; an underdetermined part  $E^-$ , a justdetermined part  $E^0$ , and an overdetermined part  $E^+$ . This decomposition is illustrated in Figure 3 where the shaded parts correspond to non-zero elements. The overdetermined part of the model is the set of equations that contain redundancy and therefore is useful for diagnosis. A set of equations with the property  $E^+ = E$  is called a proper structurally overdetermined (PSO) set. The MSO sets defined in Section 2.1 are PSO sets with redundancy  $\varphi(E) = 1$ . The following characterization of a PSO set will be useful.

*Lemma 1.* A PSO set is a set of equations  $E$  such that  $\varphi(E)$  is positive and  $\varphi(E') < \varphi(E)$  for all proper subsets  $E'$ .

This result follows from the Dulmage-Mendelsohn decomposition, see e.g. Murota [2000].

### 5.2 Structural Conditions

The translation of the analytical conditions in Section 4 into corresponding structural conditions is given by the following result.

*Theorem 2.* Assume that  $E' : A'(z)X = b'(z)$  where  $A'$  is a generic matrix. Then conditions a)–d) in Section 4 are equivalent to the conditions:

$$\varphi(E'') \leq 0 \text{ for all } E'' \subset E' \quad (14a)$$

$$1 - n \leq \varphi(E') \leq 0 \quad (14b)$$

$$|E''| - |\text{var}(E'') \cap \mathcal{X}_2| < |E'| - |\text{var}(E') \cap \mathcal{X}_2| \\ \text{for all proper subsets } E'' \text{ of } E' \quad (14c)$$

$$\mathcal{X}_1 \subset \text{var}(E') \quad (14d)$$

**Proof.** First we show that the conditions a)–d) imply (14a)–(14d). Condition a) implies (14a), since  $A'$  can not have full row rank if there is a subset of equations with more equations than unknowns.

For a fixed  $X_1$ , the condition  $X_1 \in \mathcal{L}$  is equivalent to that system

$$A'_2 X_2 = b' - A'_1 X_1 \quad (15)$$

is consistent, where we used the partition  $A' = (A'_1, A'_2)$  and consider  $\mathcal{X}_2$ -variables as unknown variables. Condition b) implies that  $\mathcal{L}$  is a proper subset of  $\mathbb{R}^n$ . This means that (15) can not be consistent for all  $X_1$  and that the system therefore is overdetermined with respect to  $\mathcal{X}_2$ . The assumption that  $A'$  is generic implies that  $A'_2$  has a structurally overdetermined part. Furthermore, it is only this overdetermined part of the system that is used to determine if the system is consistent, and also defines  $\mathcal{L}$ , and it then follows from condition c) that the set  $E'$  has to be a PSO set with respect to the  $\mathcal{X}_2$ -variables. Condition (14c) now follows from Lemma 1.

Since  $E'$  is a PSO with respect to the  $\mathcal{X}_2$ -variables, the non-zero columns in  $A'_2$  are linearly independent. It is then straightforward to show that  $\dim \mathcal{L} = \dim \ker A'$  and then the full row-rank assumption on  $A'$  gives that

$$\dim \mathcal{L} = \dim \ker A' = |\mathcal{X}| - |E'| = -\varphi(E') \quad (16)$$

Now (14b) follows directly from condition b). Finally (14d) follows from condition d).

Now we shall show the reverse implications, i.e. that (14a) to (14d) imply condition a) to d). First, condition (14a) means that  $E'$  has no overdetermined part. For a generic matrix, this is equivalent to that  $A'$  has full row rank, i.e. that condition a) is fulfilled.

Furthermore, it follows from (14c) that the set  $E'$  is a PSO set with respect to the  $\mathcal{X}_2$ -variables. This implies that the non-zero columns in  $A'_2$  are linearly independent and then (16) holds. This, together with (14b) implies condition b).

Consider now  $E'' = E' \setminus \{e\}$  for any  $e \in E'$  which is a proper subset of  $E'$ . Let  $A''_1$ ,  $A''_2$ , and  $b''$  be the corresponding model matrices in  $E''$ . Since  $E'$  is a PSO with respect to the  $\mathcal{X}_2$  variables, it follows that  $A'_2$  has no under determined or exactly determined parts. This implies that  $A''_2$  has no under determined parts and therefore the non-zero columns in  $A''_2$  are linearly independent. As above, this implies that the affine subspace defined by

$$\{X_1 : \exists X_2, A''(z)X = b''(z)\} \quad (17)$$

has the same dimension as  $\ker A''$  which is equal to  $\dim \ker A' - 1$ . Thus,  $\mathcal{L}$  is a proper subset of the subspace (17) which implies condition c).

Finally, we show d) by contradiction. Thus, assume that  $\mathcal{L}$  can be represented as in (13) and let  $c$  be the  $j$ :th column in  $A'_1$ . Then it holds that  $c \in \text{Im } A'_2$  and it follows from (14d) that  $c \neq 0$ . This implies that  $[A'_2 \ c]$  has a non-empty null space and a consequence is then that  $[A'_2 \ c]$  has an

under determined part. This implies that  $A'_2$  has a non-empty exactly determined part which contradicts (14c). Hence, condition d) must be true and this completes the proof.  $\square$

## 6. STRUCTURAL ALGORITHM

Now, a structural algorithm is derived that finds all minimal sets of equations that fulfill the conditions (14a) to (14d). The result is an algorithm that computes the set M-TestSet introduced in Section 3.

The method is based on the approach in Krysander et al. [2008] where an efficient method is developed to find all MSO sets in a structural model. The algorithm is a top-down approach that traverses all PSO sets in the model and it is therefore straightforward to modify the MSO algorithm to find all PSO sets with a specified redundancy.

### 6.1 Reformulation of the problem

The objective is now to transfer the problem of finding sets that fulfills the conditions in Theorem 2 to a problem where the PSO algorithm can be used. This can be done in the following way. First introduce the fictitious set of equations

$$E_{\mathcal{M}} = \{e_{\mathcal{M}1}, \dots, e_{\mathcal{M}n}\}$$

with the diagonal structure

$$\text{var}(e_{\mathcal{M}i}) = \{x_{1i}\}$$

Now consider the structure model of the set  $\tilde{E} = E' \cup E_{\mathcal{M}}$ . The key result that links the conditions in Theorem 2 to the PSO algorithm is the following:

*Theorem 3.* The set  $E'$  fulfills the structural characterization (14a)–(14d) if and only if and  $\tilde{E}$  fulfills the conditions:

$$\varphi(E'') \leq 0 \text{ for all } E'' \subset E' \quad (18a)$$

$$1 \leq \varphi(\tilde{E}) \leq n \quad (18b)$$

$$\tilde{E} \text{ is PSO set} \quad (18c)$$

**Proof.** First, we note that (14a) and (18a) are identical. Assume now that that  $E'$  fulfills (14b)–(14d). Then  $\text{var}(\tilde{E}) = \text{var}(E')$  and

$\varphi(\tilde{E}) = |\tilde{E}| - |\text{var}(\tilde{E})| = |E'| + n - |\text{var}(E')| = \varphi(E') + n$  and (18b) follows from (14b).

Now, we show that  $\tilde{E}$  is a PSO set. Assume that  $\tilde{E}'' = E'' \cup E''_{\mathcal{M}}$  is proper subset of  $\tilde{E}$ , where  $E'' \subset E'$  and  $E''_{\mathcal{M}} \subset E_{\mathcal{M}}$ . First, consider the case where  $E''_{\mathcal{M}}$  is a proper subset of  $E_{\mathcal{M}}$  and  $E''$  is not a proper subset of  $E'$ . Then (14d) gives that  $\mathcal{X}_1 \subset \text{var}(\tilde{E}'') = \text{var}(\tilde{E})$  and it follows that

$$\varphi(\tilde{E}'') < \varphi(\tilde{E})$$

Now, assume that  $E''$  is a proper subset of  $E'$ . It then follows from (14c) and  $|E'_{\mathcal{M}}| = |\mathcal{X}_1|$  that

$$\varphi(\tilde{E}'') = |E''| + |E''_{\mathcal{M}}| - |\text{var}(E'' \cup E''_{\mathcal{M}})| \\ = |E''_{\mathcal{M}}| - |\text{var}(E'' \cup E''_{\mathcal{M}}) \cap \mathcal{X}_1| \\ + |E''| - |\text{var}(E'' \cup E''_{\mathcal{M}}) \cap \mathcal{X}_2|$$

Observe that

$$|E''_{\mathcal{M}}| - |\text{var}(E'' \cup E''_{\mathcal{M}}) \cap \mathcal{X}_1| \leq |E''_{\mathcal{M}}| - |\text{var}(E''_{\mathcal{M}})| = 0$$

and that

$$|E''| - |\text{var}(E'' \cup E''_{\mathcal{M}}) \cap \mathcal{X}_2| = |E''| - |\text{var}(E'') \cap \mathcal{X}_2| \\ < |E'| - |\text{var}(E') \cap \mathcal{X}_2| = |\tilde{E}| - |\text{var}(\tilde{E})| = \varphi(\tilde{E})$$

where the inequality is due to (14c). Hence  $\varphi(\tilde{E}'') < \varphi(\tilde{E})$  for all proper subsets  $E''$  of  $E'$  and it follows that  $\tilde{E}$  is a PSO set. We have shown that  $\tilde{E}$  fulfills (18b) and (18c).

Now we show the reverse implication and assume that  $\tilde{E} = E' \cup E_{\mathcal{M}}$  fulfills (18b) and (18c). First we show (14d) by contradiction. Assume that  $\text{var}(E') \cap \mathcal{X}_1 = \mathcal{X}'_1$  is a proper subset of  $\mathcal{X}_1$  and let  $E'_{\mathcal{M}}$  be the corresponding diagonal structure with  $\text{var}(E'_{\mathcal{M}}) = \mathcal{X}'_1$ . Observe that

$$\begin{aligned} |E'_{\mathcal{M}}| &= |\text{var}(E'_{\mathcal{M}} \cup E') \cap \mathcal{X}_1| = |\mathcal{X}'_1| \\ |E_{\mathcal{M}}| &= |\text{var}(E_{\mathcal{M}} \cup E') \cap \mathcal{X}_1| = |\mathcal{X}_1| \end{aligned}$$

It then holds that

$$\begin{aligned} \varphi(E'_{\mathcal{M}} \cup E') &= |E'_{\mathcal{M}}| - |\text{var}(E'_{\mathcal{M}} \cup E') \cap \mathcal{X}_1| \\ &\quad + |E'| - |\text{var}(E'_{\mathcal{M}} \cup E') \cap \mathcal{X}_2| \\ &= |E_{\mathcal{M}}| - |\text{var}(E_{\mathcal{M}} \cup E') \cap \mathcal{X}_1| \\ &\quad + |E'| - |\text{var}(E_{\mathcal{M}} \cup E') \cap \mathcal{X}_2| \\ &= \varphi(E_{\mathcal{M}} \cup E') = \varphi(\tilde{E}) \end{aligned}$$

which contradicts that  $\tilde{E}$  is a PSO set.

Furthermore, condition (14d) implies that  $\text{var}(E' \cup E_{\mathcal{M}}) = \text{var}(E')$  and then

$$\varphi(\tilde{E}) = |E'| + |E_{\mathcal{M}}| - |\text{var}(E' \cup E_{\mathcal{M}})| = \varphi(E') + n$$

and condition (14b) now follows directly from (18b).

Finally, we show inequality (14c) by contradiction. Assume that

$$|E''| - |\text{var}(E'' \cap \mathcal{X}_2)| \geq |E'| - |\text{var}(E') \cap \mathcal{X}_2|$$

for some proper subset  $E''$ . Let  $\tilde{E}'' = E_{\mathcal{M}} \cup E''$ , which is a proper subset of  $\tilde{E}$ . By using the assumption we get

$$\begin{aligned} \varphi(\tilde{E}'') &= |E_{\mathcal{M}}| + |E''| - |\text{var}(E_{\mathcal{M}} \cup E'')| \\ &= |E_{\mathcal{M}}| - |\mathcal{X}_1| + |E''| - |\text{var}(E'' \cap \mathcal{X}_2)| \\ &\geq |E_{\mathcal{M}}| - |\mathcal{X}_1| + |E'| - |\text{var}(E') \cap \mathcal{X}_2| = \varphi(\tilde{E}) \end{aligned}$$

which contradicts that  $\tilde{E}$  is a PSO set, i.e. condition (18c).  $\square$

## 6.2 Algorithm to compute M-TestSet

To compute the previously defined set  $\text{M-TestSet}(\mathcal{M}, E)$ , we shall use a modified version  $\text{FindPSO}(E, F, n)$  of the MSO algorithm in Krysander et al. [2008]. The algorithm computes all PSO sets  $E' \subset E$  that fulfill the additional conditions  $F \subset E'$  and  $\varphi(E') \leq n$ . All sets that fulfill the conditions (18b) and (18c) are found by the algorithm  $\text{FindPSO}(E, \text{var}(\mathcal{M}), |\text{var}(\mathcal{M})|)$ . After that, the sets that do not fulfill condition (18a) are removed and this is equivalent to removing sets where  $E^+ \neq \emptyset$ . The sets that remains is the set  $\text{M-TestSet}(\mathcal{M}, E)$ .

## 7. APPLICATION OF THE METHOD ON A DYNAMIC MODEL

Now we return to the dynamic example introduced in Section 2.1. The algorithm was initially developed to handle static problems. For example, the test (1) can not be derived without considering the dynamical behavior of the system. Now it will be shown how the algorithm also can be used to analyze dynamic models.

The dynamical problem is transferred to a static problem by considering unknown signals and differentiated

unknown signals as separate independent signals. By differentiating equations, new useful relations are obtained. For example, by differentiating the static equations  $e_4, e_5, e_6, e_7$  in the tank model we obtain the set of equations  $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, \dot{e}_4, \dot{e}_5, \dot{e}_6, \dot{e}_7\}$ . The structural model for this set is shown in Figure 4.

equation	unknown X									
	$x_1$	$x_2$	$x_3$	$q_1$	$q_2$	$\dot{x}_1$	$\dot{x}_2$	$\dot{x}_3$	$\dot{q}_1$	$\dot{q}_2$
$e_1$				X		X				
$e_2$				X	X		X			
$e_3$					X			X		
$e_4$	X	X		X						
$e_5$		X	X		X					
$e_6$	X									
$e_7$				X						
$\dot{e}_4$						X	X		X	
$\dot{e}_5$							X	X		X
$\dot{e}_6$							X			
$\dot{e}_7$									X	

Fig. 4. Structural description of the differentiated model.

Running the algorithm on this structural model, together with constraints  $c_1, c_2, c_3$ , 18 sets are found. Some tests are obvious, like the tests presented in Section 2.1, but others are more complex, for example the set  $\{e_2, e_4, e_5, e_6, \dot{e}_4, \dot{e}_6, \dot{e}_7, c_2, c_3\}$  for which the test  $\dot{y}_1 - \dot{y}_2 - y_1 \in [-2h_2, h_3]$  can be derived.

## 8. CONCLUSIONS

Structural analysis has been proven to be a useful tool when deriving analytical redundancy relations for fault diagnosis. This work extends structural analysis to include models with more general relations, for example inequality constraints on state variables. By considering this extended class of models, and also extending the class of redundancy relations, better fault isolation performance can be obtained.

Key results of the paper are an analytical and a structural characterization of sets of equations and constraints that can be used to derive tests. An algorithm has been developed that finds all submodels that fulfill the structural characterization. The method is demonstrated on a differential-algebraic model.

## REFERENCES

- D. Arnon. A bibliography of quantifier elimination for real closed fields. *Journal of symbolic computation*, 5(1-2): 267–274, 1988.
- M. Blanke, M. Kinneart, J. Lunze, and M. Staroswiecki. *Diagnosis and Fault-Tolerant Control*. Springer-Verlag, 2003.
- D. Cox, J. Little, and D. O’Shea. *Ideals, Varieties, and Algorithms*. Springer-Verlag, 1991.
- Mats Jirstrand. Nonlinear control system design by quantifier elimination. *Journal of Symbolic Computation*, 24(2):137–152, August 1997.
- Mattias Krysander, Jan Åslund, and Mattias Nyberg. An efficient algorithm for finding minimal over-constrained sub-systems for model-based diagnosis. *IEEE Transactions on Systems, Man, and Cybernetics – Part A: Systems and Humans*, 38(1), 2008.
- K. Murota. *Matrices and Matroids for System Analysis*. Springer-Verlag, 2000. ISBN 3-540-66024-0.
- B. Pulido and C.A. Gonzalez. Possible conflicts: a compilation technique for consistency-based diagnosis. *IEEE Transactions on Systems Man and Cybernetics – Part B*, 34(5):2192–2206, October 2004.