Nonlinear observers

basic notions and linear systems

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This course

Objectives

Course aimed at PhD students that are not doing research in estimation.

1. Basic definitions, fundamentals
2. General knowledge on design methodologies

Examination

- Compulsary attendance on lectures
- Solve course exercises and active participation in discussions

Course organization

- Lectures as a complement to books/papers. You can not take this course by only attending lectures, reading is a must!
- 6 credits
- prel. 4 lectures, 4 discussions

This lecture

- Fundamentals for observers for linear systems
- Not much about design today
- An overview and possibly a different perspective than Kailath.
- Text: Kailath Chapter 2.3, 2.4 and Chapter 4 + additional notes.
- Exercises:
  - Kailath: 2.3-1a, 2.3-3, 2.3-5, 2.3-15, 2.3-16, 2.4-2a, 2.4-4, 2.4-5a, 2.4-8a, 4.1-3, 4.1-7, 4.1-8, 4.1-9, 4.2-2, 4.3-3, 4.3-4
  - The marked exercises in this lecture Le1.1 - Le1.6
  - Exercises 2.3-22, 2.4-6, 4.1-4, 4.2-2, 4.2-3 are interesting (requirec last time, but not mandatory this time. But do take a look!)

Outline

What is an observer?

What is an observer, stateobservers, filter, Kalman filter, Luenberger observer ...

$$\begin{aligned} \text{Model :} \quad & \dot{x} = f(x,u) \\ & y = h(x) \\ \text{Observer :} \quad & \dot{w} = g_1(w,y,u) \\ & \hat{x} = g_2(w,y,u) \end{aligned}$$

You want $\hat{x}(t) = x(t)$, or minimal variance or

$$\lim_{t\to\infty} \hat{x}(t) = x(t)$$

Typical case

For a model

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned}$$

Typical observer:

$$\dot{\hat{x}} = f(\hat{x},u) + K(\hat{x},y,\dots)(y - h(\hat{x}))$$

where the observer gain $K()$ is chosen such that the error dynamics is stable, i.e., $e(t) = x(t) - \hat{x}(t)$

$$\lim_{t\to\infty} e(t) = 0$$

What is the usual way?

$$\begin{aligned} \dot{x} &= Ax + Bu & \qquad \dot{\hat{x}} &= A\hat{x} + Bu + K(y - C\hat{x}) \\ y &= Cx \end{aligned}$$

with the estimation error $e = x - \hat{x}$ the error dynamics is

$$\dot{e} = (A - KC)e$$

- why is the feedback necessary? Systems are often stable by themselves?
- how do you choose K and what are your objectives?
- when can we choose K so that we get desired properties?

Observer as a parameter estimator

Assume a parameterized, by θ, function

$$y_t = h(u_t;\theta)$$

and we have measured data y and u and are interested in estimating the constant θ.
This can be written as an estimation problem for the state-space form

$$\begin{aligned} \theta_{t+1} &= \theta_t \\ y_t &= h(u_t;\theta) \end{aligned}$$

where the state is the parameters we are interested in.

Observer as a parameter estimator, cont.

If the system is dynamic, it is equally "easy". Rewrite the model

$$
\begin{aligned}
x_{t+1} &= f(x_t, u_t; \theta) \\
y_t &= h(x_t, u_t; \theta)
\end{aligned}
$$

as

$$
\begin{aligned}
\theta_{t+1} &= \theta_t \\
x_{t+1} &= f(x_t, u_t; \theta) \\
y_t &= h(x_t, u_t; \theta)
\end{aligned}
$$

and estimate the new, extended, state. This means that the state x in the original model is estimated simultaneously as the parameter state θ.

A word of warning

Note!

The is certainly not the whole truth about how to estimate parameters in nonlinear dynamical systems. There are many traps and pitfalls.

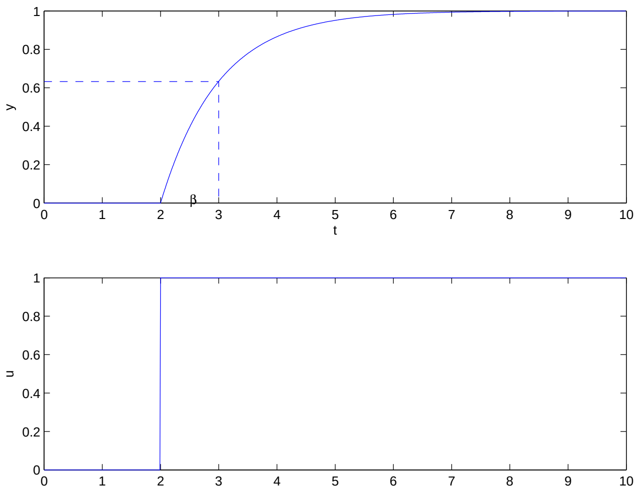
Paper (linked from the course page) that analyses the problem

Lennart Ljung, "*Asymptotic behavior of the extended Kalman filter as a parameter estimator for linear systems*", IEEE Transactions on Automatic Control, vol. 24, no. 1, pp. 36-50, 1979.

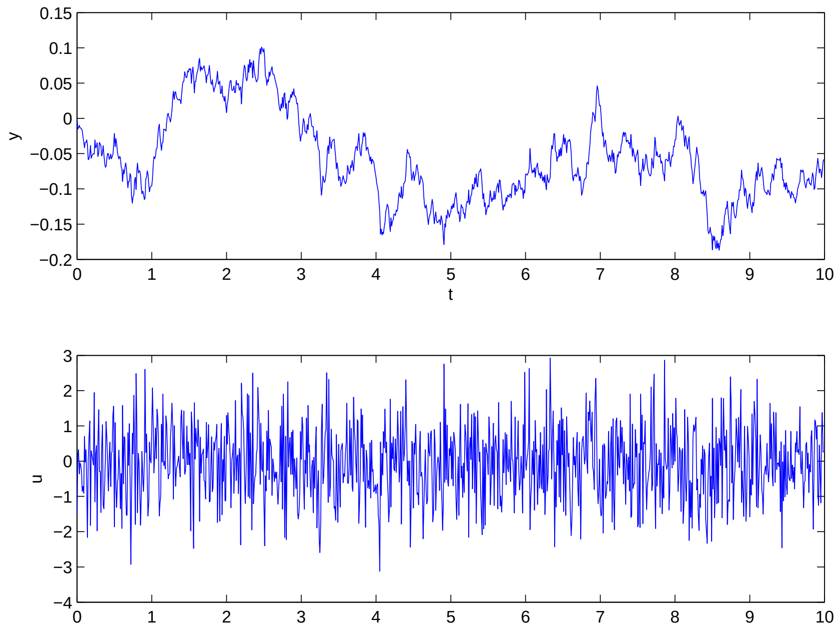
Observer as a parameter estimator, example

Assume a first order system and we are interested in the parameter β

$$
\begin{aligned}
\dot{x} &= -\beta x + u \\
y &= x
\end{aligned}
$$



A more general input signal, what then?



Step response experiment

Discretize the system with a basic euler forward

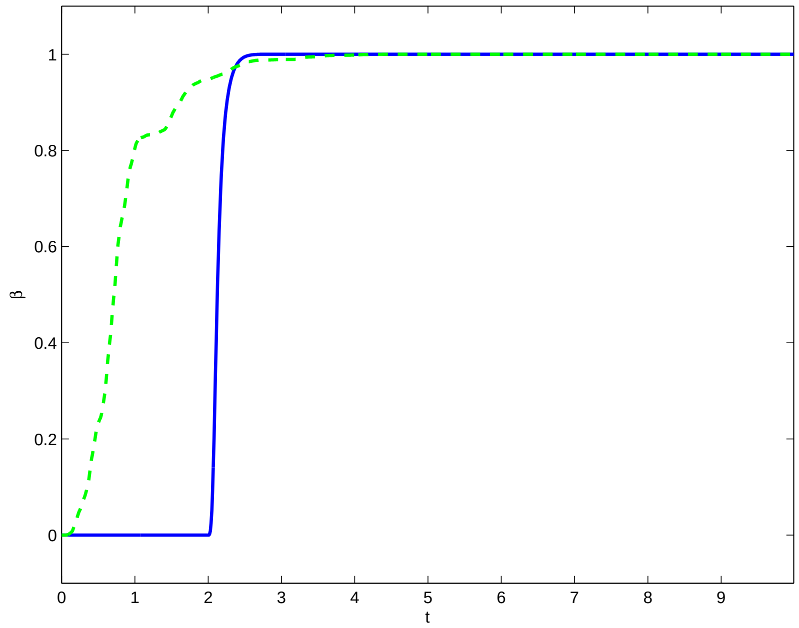
$$\beta_{t+1} = \beta_t$$
$$\frac{x_{t+1} - x_t}{T_s} = -\beta_t x_t + u_t$$
$$y_t = x_t$$

Now we have a regular state-space form and if we can estimate the state vector we have also estimated the parameter β.
designing an Extended Kalman Filter (EKF)

$$\hat{\beta}_{t+1} = \hat{\beta}_t \qquad +K_t^1(y_t - \hat{x}_t)$$
$$\hat{x}_{t+1} = \hat{x}_t - T_s\hat{\beta}_t\hat{x}_t + T_s u_t \qquad +K_t^2(y_t - \hat{x}_t)$$

It is sensitive how to determine the observer gains K_t^1 and K_t^2. More about that later.

EKF as a recursive parameter estimator



When does it work?

Example 1:

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -x_2 + u$$
$$y_1 = x_1$$
$$y_2 = x_2$$

Does it work with only y_1?
With only y_2?

Example 2:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 + u$$
$$y = x_1 + x_2$$

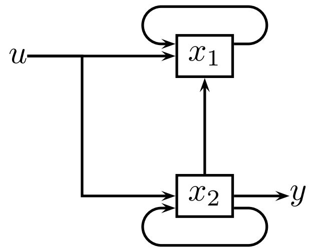
What now?

To be able to design a stable observer, the non-measured signals must be **sufficiently** visible in the measurements.

Partitioning

Basically, a non-observable system can always be written in the form

$$\dot{x}_1 = g_1(x_1, x_2)$$
$$\dot{x}_2 = g_2(x_2)$$
$$y = h(x_2)$$



Outline

Observer, definition

There are different version in the litterature, often similar to

Assume an autonomous dynamical system

$$\dot{x} = f(x), \quad y = h(x) \tag{1}$$

Definition (Observer)

A dynamical system

$$\dot{z} = \Phi(z, y)$$

is an observer for (1) if there is a (locally) invertible function $T(x)$ such that

$$T(x(0)) = z(0) \Rightarrow z(t) = T(x(t))$$

Possibly you can add a stability requirement in the definition.

Observability - when does it work?

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0$$
$$y = Cx$$

Definition (Observability)

A linear system is said to be observable (on $[0, t_f]$) if x_0 is uniquely determined by $y(t)$ ($t \in [0, t_f]$).

Definition (Observability)

The state $x_0 \neq 0$ is said to be non-observable if when $u(t) = 0, \; t \geq 0$ the output is $y(t) = 0, \; t \geq 0$.

Observability

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0$$
$$y = Cx$$

Definition (Observability)

A system is observable if the observability matrix

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

has full column rank.

Observability

$$\dot{x} = Ax + Bu, \;\; x(0) = x_0$$
$$y = Cx$$

Definition (Obsaervability)

A system is observable if the observability gramian

$$\Sigma_o = \int_0^{\infty} e^{A^T\tau} C^T C e^{A\tau} d\tau$$

is invertible.

(I'll get back to this later)

Observability

My favourites are variants of the ones below

Definition (Observability)

Let $y_1(t)$ be the output when $x(0) = x_1$ and $y_2(t)$ the output when $x(0) = x_2$. A linear system is observable if $y_1(t) = y_2(t) \Rightarrow x_1 = x_2$.

Definition (Observability)

A linear system is observable (on $[0, t_f]$) if $x(t)$ ($t \in [0, t_f]$) is uniquely determined by $y(t)$ ($t \in [0, t_f]$).

Both these definitions are directly extendable to non-linear and time-varying systems.

Observer by differentiation

A "naïve" method:

$$\dot{x} = Ax, \quad y = Cx$$

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(\rho-1)}(t) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{\rho-1} \end{pmatrix} x(t) = \mathcal{O}x(t)$$

- observability = possible to solve the system of equations
- observability index
- can $\rho < n$, $\rho > n$, always $\rho = n$?
- analogous in discrete time

Observer by differentiation, cont.

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y^{(\rho_1-1)}(t) \\ \vdots \\ y_2(t) \\ \vdots \\ y^{(\rho_2-1)}(t) \\ \vdots \end{pmatrix} = \begin{pmatrix} C_1 \\ \vdots \\ C_1A^{\rho_1-1} \\ \vdots \\ C_2 \\ \vdots \\ C_2A^{\rho_2-1} \\ \vdots \end{pmatrix} x(t)$$

- not necessarily the case that you have to differentiate all measurement signals equally many times
- minimal ρ_i are called the observability indices
- For an observable system, $\max \rho_i =?$, $\sum \rho_i =?$

Observerbality indices

It is not necessary to differentiate all measurement signals equally many times for $\mathcal{O}$ to have full column rank

	c_1	c_2	c_3
A^0	x	x	x
A^1	x		x
A^2	x		
A^3			

$$\text{rank} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_1 A \\ c_3 A \\ c_1 A^2 \end{bmatrix} = 6, \qquad \begin{pmatrix} y_1 \\ \dot{y}_1 \\ \ddot{y}_1 \\ y_2 \\ y_3 \\ \dot{y}_3 \end{pmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_1 A \\ c_3 A \\ c_1 A^2 \end{bmatrix} x$$

The lengths of the "chains" $\{\rho_i\} = \{3, 1, 2\}$ are the observerbability indices. Realisation theory and how "deep" you have to dig to access the true state value.

Asymptotic observers

If (C, A) is an observable pair the poles of $A - KC$ can be placed arbitrarily

- An asymptotic observer fulfills the condition

$$\lim_{t\to\infty} x(t) - \hat{x}(t) = 0$$

For the model and the observer

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad & \dot{\hat{x}} &= A\hat{x} + Bu + K(y - C\hat{x}) \\ y &= Cx + Du \end{aligned}$$

the error dynamics is $(e = x - \hat{x})$ $\dot{e} = (A - KC)e$
The eigenvalues for för $A - KC$ can be places arbitrarily with a well chosen K if and only if (A, C) is an observable pair.

Design choices

- more than 1 output $\Rightarrow$ more freedom to place the poles
- place the poles in -1 and -2, what converges as e^{-2t} and what converges as e^{-t}?
- modes of the error dynamics (eigenstructure assignment)
- noise and minimal variance (Kalman-filter)
- robustness agains modelling errors in different norms and criteria
  - parametric uncertainty
  - guarantee area for the poles under uncertainty
  - ...

Ackermann's formula

- One (among many) known direct formula for controllable SISO systems
- Bad numerical properties
- See course material for a derivation of the formula

$$K = \alpha_c(A)\mathcal{O}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \alpha_c(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_0 I$$

and α_i are the coefficients in the *desired* polynomial

$$\det \lambda I - A + KC = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0$$

Observable canonical form

Why is it called observable canonical form?

$$y(t) = \frac{b(p)}{a(p)}u(t) = \frac{b_1 p^{n-1} + \cdots + a_{n-1}p + b_n}{p^n + a_1 p^{n-1} + \cdots + a_{n-1}p + a_n}u(t)$$

$$\dot{x} = \begin{pmatrix} -a_1 & 1 & 0 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ -a_{n-1} & 0 & 0 & \dots & 0 & 1 \\ -a_n & 0 & 0 & \dots & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} x$$

$$A - KC = \begin{pmatrix} -(a_1 + k_1) & 1 & 0 & 0 & \dots & 0 \\ -(a_2 + k_2) & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ -(a_{n-1} + k_{n-1}) & 0 & 0 & \dots & 0 & 1 \\ -(a_n + k_n) & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Nonlinear canonical form

More on this later, but a direct non-linear counterpart is

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} x + \begin{pmatrix} f_1(x_1, u) \\ f_2(x_1, u) \\ \vdots \\ f_{n-1}(x_1, u) \\ f_n(x_1, u) \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} x$$

If you can translate a system into this form you can accomplish linear error dynamics. More on this later.

Observable realizations

$$y(t) = \frac{b(p)}{a(p)}u(t) \Leftrightarrow a(p)y(t) - b(p)u(t) = 0$$

- can above model correspond to a non-observable system? Does it depend on if $a(s)$ and $b(s)$ has common zeros?
- can it be realizrd by a non-observable state-space model?

Observable realizations

Realize

$$y(t) = \frac{p+1}{p^2 + 3p + 2}u(t)$$

in controllable and observable canonical forms respectively

$$\dot{x} = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad \dot{x} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

- both these realizations, do they correspond to the same system?
- is it possible to find a state transformation from one to the other?
- explain

Summary

This far

- What is an observer?
- Observability, when does it work? Definition for linear systems.
- Observervability index, observability indices
- Asymptotic observer
- Canonical forms for linear systems, observable realizations

Outline

Based on the observability matrix

$$\dot{x} = Ax, \quad y = Cx$$

Theorem (Observability)

The pair (C, A) is observable if and only if

$$\text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$$

- simple
- numerically not so good for large (and not so large) systems

PBH rank/eigenvector test

$$\dot{x} = Ax, \quad y = Cx, \text{ can be written as } \begin{pmatrix} C \\ \lambda I - A \end{pmatrix} x = 0$$

Theorem (PBH eigenvector)

The pair (C, A) is non-observable if and only if there exists an $x \neq 0$ such that

$$Ax = \lambda x, \quad Cx = 0$$

Theorem (PBH rank)

The pair (C, A) is observable if and only if

$$\begin{pmatrix} C \\ \lambda I - A \end{pmatrix}$$

has full column rank for all $s \in \mathbb{C}$.

really useful, connect to geometric interpretation.

PBH

Sketch proof for

$$\exists x.\ \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x = 0 \Leftrightarrow \exists \lambda, x.\ \begin{pmatrix} C \\ \lambda I - A \end{pmatrix} x = 0$$

$\Leftarrow$ Direct substitution

$\Rightarrow$ Without loss of generality, assume system is partitioned in an observable and non-observable part, i.e.,

$$\dot{x} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} x + Bu$$
$$y = \begin{pmatrix} 0 & C_2 \end{pmatrix} x$$

where (C_2, A_{22}) is an observable pair.

Exercise: non-observable subspace

Assume a non-observable model

$$\dot{x} = Ax, \ \ y = Cx$$

The null-space for the observability matrix $\mathcal{O}$ gives the non-pobservable subspace. This means that there exists λ and v such that

$$\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} v = 0$$

Exercise Le1.1: What is the relation between v in the null-space above to those in the non-observable subspace? Is the null-space equal to the non-observable subspace?
Hint: Jordan forms.

Generalized observability matrix

We are going to derive a condition that is numerically more attractive than direct application of $\mathcal{O}$.
Differentiate the state-space model as before, but do not substitute

$$\begin{array}{llll} \dot{x} = Ax & \ddot{x} = A\dot{x} & \dots & x^{(n-1)} = Ax^{(n-2)} \\ y = Cx & \dot{y} = C\dot{x} & \dots & y^{(n-1)} = Cx^{(n-1)} \end{array}$$

Collect all the x and y you get the system of equations

$$\left(\begin{array}{cccccc} I & -A & 0 & \dots & 0 & 0 \\ 0 & I & -A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & -A \\ \hline 0 & 0 & 0 & \dots & 0 & C \\ 0 & 0 & 0 & \dots & C & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & C & 0 & \dots & 0 & 0 \\ C & 0 & 0 & \dots & 0 & 0 \end{array}\right) \left(\begin{array}{c} x^{(n-1)} \\ x^{(n-2)} \\ \vdots \\ \\ \hline \vdots \\ \\ \dot{x} \\ x \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \hline y \\ \dot{y} \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{array}\right)$$

Generalized observability matrix

You can show that the system is observable if and only if $\mathcal{O}_{ext}$ has full column rank

$$\mathcal{O}_{ext} = \begin{pmatrix} I & -A & 0 & \dots & 0 & 0 \\ 0 & I & -A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & -A \\ 0 & 0 & 0 & \dots & 0 & C \\ 0 & 0 & 0 & \dots & C & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & C & 0 & \dots & 0 & 0 \\ C & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(n(n+m)-n)\times n^2}$$

$\mathcal{O}_{ext}$ nice enough to allow structural analysis, which $\mathcal{O}$ does not allow.

Observability gramian

The observability gramian was mentioned earlier.
We can write

$$y(t) = Ce^{At}x_0 \Rightarrow (e^{At})^T C^T y(t) = (e^{At})^T C^T Ce^{At}x_0$$
$$\Rightarrow \int_0^\infty (e^{At})^T C^T y(t)dt = \int_0^\infty (e^{At})^T C^T Ce^{At}dtx_0 = \Sigma_o x_0$$

and then derive the result

Theorem (Observability)

An LTI system is observable if, and only if, the observability gramian

$$\Sigma_o = \int_0^\infty e^{A^T\tau} C^T Ce^{A\tau} d\tau$$

is invertible.

Time variable systems

$$\dot{x}(t) = A(t)x(t), \quad y(t) = C(t)x(t),\ x(0) = x_0$$

where $A(t)$ and $C(t)$ are continuous matrix functions. Whit

$$y(t) = C(t)x(t) = C(t)\Phi(t)x_0 \quad (\Phi(t) = e^{At} \text{ for LTI system})$$

As before, let

$$\Sigma_o(t) = \int_0^t \Phi^T(\tau)C^T(\tau)C(\tau)\Phi(\tau)d\tau$$

define the observability gramian $\Sigma_o(t)$.

The system is observable on $[0, t]$ if there is any t in the interval such that $\Sigma_o(t)$ is invertible.

There are similar results for $\mathcal{O}$, but more on that in the nonlibear part of the course.

Exercise on time variable systems

Exercise Le1.2: Assume a linear system with a function $f()$ according to

$$\dot{x} = -ax + f(x; \theta_1, \theta_2)$$
$$y = x$$

and we are interested in estimating x and the function parameters θ_i with an observer. Assume that the function f is linear, i.e.,

$$f(x; \theta_1, \theta_2) = \theta_1 + (\theta_2 - \theta_1)x$$

The state x is observable (we are measureing it directly), rewrite the system as

$$\dot{\theta}_1 = 0, \quad \dot{\theta}_2 = 0 \tag{2a}$$
$$z = \dot{y} + ay = \begin{pmatrix} 1-y & y \end{pmatrix} \theta = C(t)\theta \tag{2b}$$

We have now a time-variable linear system. Use the gramian criterion to show that the system (2) is observable on the interval $[0, t]$, $t > 0$ if $\dot{y} \neq 0$. Why is this result expected?

Summary

Test to verify observablity for linear systems based on

- observability matrix
- PBH tests
- Generalized observability matrix
- Observability gramian, which also works for time varying systems

Outline

Is observability sufficient?

Observability tells us if it is possible to find x_0 given $y(t)$, but it doesn't tell us how *much* of x_0 that is visible in the output. For a stable and observable system, the energy of $y(t)$ is one measure.

$$\dot{x} = Ax,\ x(0) = x_0, \qquad y = Cx \qquad \int_0^\infty y(\tau)^T y(\tau) d\tau = \cdots = x_0^T \Sigma_o x_0$$

Exercise Le1.3: Show that the observability gramian Σ_o satisfies the Lyapunov equation

$$A^T\Sigma_o + \Sigma_o A + C^T C = 0$$

Indicate where the stability requirement is used.

Hints:

1. Matrices e^{At} and A commutates
2. $\frac{d}{dt}e^{A^T t}Me^{At}$ is an interesting expression.

Further, the symmetrical solution to the Lyapunov equation is unique under pretty general conditions.

Observerbarhetsgramianen

$$\dot{x} = \begin{pmatrix} -2 & -1 \\ \epsilon & -1 \end{pmatrix} x$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

Exercise Le1.4: Compute the observability gramian as a function of ϵ and interpret the result.

Outline

The non-observable subspace

The matrix $\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$ seems important.

- non observable subspace
- why can't you speak of the observable subspace

Example:

$$\dot{x} = \begin{pmatrix} -2 & -1 \\ 0 & -1 \end{pmatrix} x, \quad y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

$$\mathcal{O} = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}, \quad \ker \mathcal{O} = \operatorname{Im} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This means that all $x(t)$ on the line

$$x(t) = x_0 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} s$$

have the same effect on $y(t), \dot{y}(t), \ldots$.

Geometrical interpretations

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

Take a subspace $\mathcal{V}$ that fulfills

a) $A\mathcal{V} \subseteq \mathcal{V}$ ($\mathcal{V}$ is A invariant)

b) $\mathcal{V} \subseteq \ker C$ ($\mathcal{V}$ is in the null-space of matrix C)

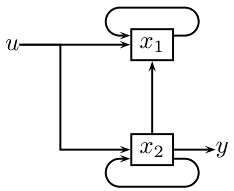
c) $\mathcal{V}$ is the largest linear space that fulfills conditions a and b

How do we compute $\mathcal{V}$? What does $\mathcal{V}$ mean?
Nonlinear generalizations exists.

Decomposition

The last slide gives us that, by a change of variables, the model can be written in a form where x_1 is non-observable and x_2 observable

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} x + Bu \\ y &= \begin{pmatrix} 0 & C_2 \end{pmatrix} x \end{aligned}$$



Detectability

$$\dot{x} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} x, \quad y = \begin{pmatrix} 0 & C_2 \end{pmatrix} x$$

the pair (A_{22}, C_2) is observable

$$A - KC = \begin{pmatrix} A_{11} & A_{12} - K_1 C_2 \\ 0 & A_{22} - K_2 C_2 \end{pmatrix}$$

$$\begin{aligned} e_2(t) &= e^{(A_{22} - K_2 C_2)t} e_2(0) \\ \dot{e}_1(t) &= A_{11} e_1 + (A_{12} - K_1 C_2) e_2 \end{aligned}$$

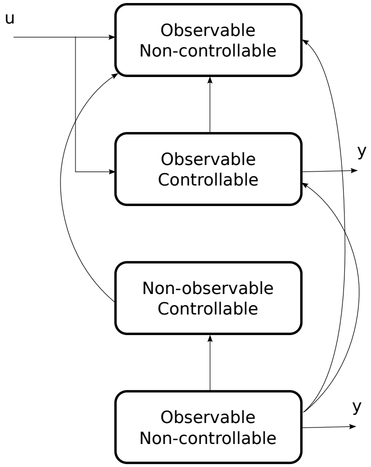
The existance of an asymptotic observer does not require observability, only detectability (all non-observable modes are stable).

Kalman decomposition

More general:

$$\dot{x} = \begin{pmatrix} A_{c\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{co} & 0 & A_{24} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{c}o} \end{pmatrix} x + \begin{pmatrix} B_{c\bar{o}} \\ B_{co} \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & C_{co} & 0 & C_{\bar{c}o} \end{pmatrix}$$



Outline

Reduced observers

$$\dot{x} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x, \quad y = x_1$$

- the observer normally has the same order as the system
- We are measuring some states directly, this can be utilized
- We can design an observer with order $n - n_1$. We gain something, what do we lose?

$$\hat{x}_1 = y, \quad \dot{\hat{x}}_2 = ?$$

Reduced observers

There are many ways to derive reduced obsevers, I like this one where the first dynamic equation is used in the feedback (why can't the measurement equation be used?):

$$\dot{\hat{x}}_2 = A_{21}y + A_{22}\hat{x}_2 + K(\dot{y} - A_{11}y - A_{12}\hat{x}_2)$$

The only problem is that there is an $\dot{y}$ in the equation. With a new state $w = \hat{x}_2 - Ky$ we get the state-space equation for the observer

$$\begin{aligned} \dot{w} &= (A_{22} - KA_{12})w + (A_{21} + A_{22}K - KA_{11} - KA_{12}K)y \\ \hat{x}_1 &= y \\ \hat{x}_2 &= w + Ky \end{aligned}$$

- What are the error dynamics?
- What is the observability requirements for a reduced observer?
- **Exercise Le1.5**: Show that the observability condition for the original system implies that the poles can be selected arbitrarily in the error dynamics of the reduced observer.

Reduced observer

Exercise Le1.6: Design a reduced and a full-order observer for the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$$
$$y = x_1$$

Place the poles similar in the two and compare sensitivity to measurement noise in both designs.

Outline

Separation principle

Assume a linear system and state feedback from estimated states

$$\dot{x} = Ax + Bu \qquad \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$
$$y = Cx \qquad u = r - L\hat{x}$$

This gives a closed loop system with $2n$ states as

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & -BL \\ KC & A - KC - BL \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix} r$$
$$y = Cx$$

The characteristic polynomial can be shown to factorize as

$$\det(sI - A + BL)\det(sI - A + KC)$$

Separation result

Optimal controller for the system can be split into two separate optimization problems: observer and controller.

It is often the case that connecting two stable systems might give an unstable system. This is not the case here.

Separation principle

An interesting property for state feedback and observers is that the transfer function from the reference signal r to y can be shown to be

$$G_{yr}(s) = C(sI - A + BL)^{-1}B$$

which is exactly as if the observer is not there at all. Cancellations in the transfer function this way usually means non-observale/controllable states.

what this means and how it works is illustrated nby an excellent exercise in Kailath.

Observers for diagnosis

- Observers is a common way to generate residuals.
- In a residual generator you are generally not interested in the state
- additional freedom therefore appears
- fault isolation

$$\dot{x} = Ax + Bu, \qquad \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$
$$y = Cx \qquad r = y - C\hat{x}$$

Observer for diagnosis

$$\dot{x} = \begin{bmatrix} -1 & -2 \\ 3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = x$$

design a residual generator (with poles in $-\alpha$) using the observer

$$\dot{\hat{x}} = \begin{bmatrix} -1 & -2 \\ 3 & -4 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} -1+\alpha & -2 \\ 1 & -4+\alpha \end{bmatrix} (y - \hat{x})$$
$$r = y_1 - \hat{x}_1$$

Assume an actuator fault, what happens in the residual?

$$\dot{e} = (A - KC)e + Bf = \begin{pmatrix} -\alpha & 0 \\ 2 & -\alpha \end{pmatrix} e + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f_u$$
$$r = Ce = e$$

with the transfer function

$$G_{rf_u} = C(sI - A + KC)^{-1}Bf_u$$

Nonlinear observers

basic notions and linear systems

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