Vehicle Dynamics and Control

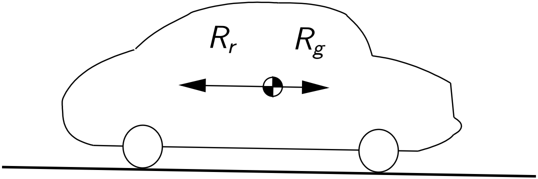
Jan Åslund
jan.aslund@liu.se
Associate Professor

Dept. Electrical Engineering
Vehicular Systems
Linköping University
Sweden

Lecture 4

Exercise 1.2 c

In this exercise the slope angle is 0.5 degrees. If the longitudinal forces acting on the car is computed carelessly, then the result looks something like this:



The resultant is pointing backwards, which is of course not correct. When using the model used previously to calculate the rolling resistance R_r, it is assumed that the vehicle is moving forward.

Conclusion:
The car is not moving at all.

Car at rest

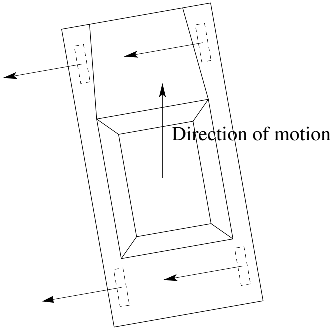
Consider a car at rest. Assume a slip based model is used for the longitudinal force when the car starts moving.

How should the slip be defined?

How can a model like the one shown in Figure 1.16 be used?

Side forces: Question 1

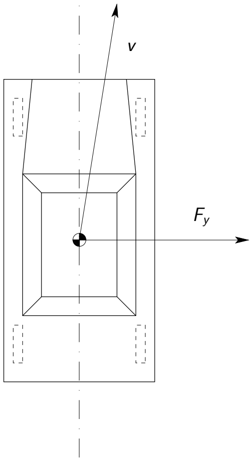
Consider a that a car moving along a straight line with constant longitudinal velocity with all wheel pointing in the direction of travel. Assume that motion of car is perturbed and the direction of the car suddenly is different from the direction of motion with no angular velocity:



Question 1: In which direction will the car turn?

Side forces: Question 2

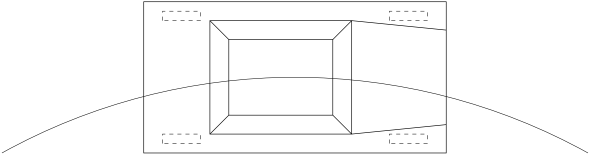
Assume that a lateral force F_y is applied at the center of gravity of a car driving straight.



Question 2: In which direction will the car turn?

Side forces: Question 3

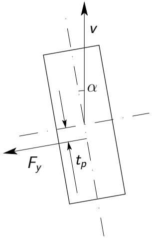
Assume that a car is taking a turn with constant speed and constant turn radius.



Question 3: Assume that the speed is increases. In which direction should the steering wheel be turned to make the car stay on the same circle?

Side forces: A simple model

A wheel moving in the lateral lateral direction give rise to a cornering force F_y och self-aligning torque $M_z = t_p \cdot F_y$:



The slip angle α is the angle between the direction of the wheel and the direction of the velocity vector **v**.

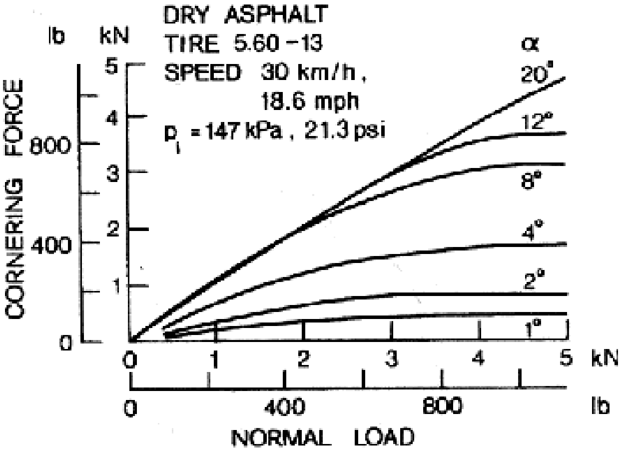
To begin with, a simple linear model will be used to represent the relation between the slip angle α and the cornering force F_y

$$F_y = C_\alpha \alpha$$

where C_α is called the cornering stiffness.

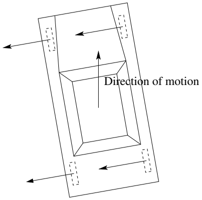
Side forces: A non-linear example

Figure 1.25 in the text book shows an example how the cornering force depends on the slip angle α and the normal force F_z:



Answer to Question 1

It was assumed that the angular velocity is zero and in this case the slip angle is the same for at four wheels.



Let denote this angle α and let the cornering stiffness of the wheel in the front and rear axis be denoted by $C_{\alpha f}$ and $C_{\alpha f}$, respectively.

The total cornering forces at the front axis and rear axis are then equal to $2C_{\alpha f}\alpha$, and $2C_{\alpha r}\alpha$, respectively.

Answer to Question 1

The moment of the cornering forces about the center of gravity is equal to

$$M_z = 2l_2 C_{\alpha r}\alpha - 2l_1 C_{\alpha f}\alpha = 2\alpha(C_{\alpha r} l_2 - C_{\alpha f} l_1)$$

with clock-wise as the positive direction.

The sign of the expression $C_{\alpha r} l_2 - C_{\alpha f} l_1$ gives the answer to Question 1.

- Positive sign: The car turns towards the direction of the unperturbed path.
- Equal to zero: The car does not turn at all and continue in the perturbed direction.
- Negative sign: The car turns counterclockwise and the deviation from the original path increases even further.

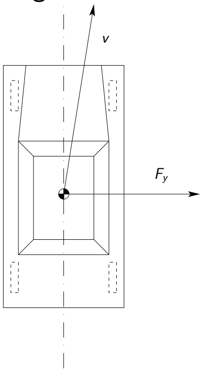
Answer to Question 1

The sign of the expression $C_{\alpha r} l_2 - C_{\alpha f} l_1$ depends on the location of he center of gravity and the ratio between the cornering stiffness in the front and rear.

Question 1 will be studied more thoroughly later in the course.

Answer to Question 2

The limit case in Question 1 was the case $C_{\alpha r} l_2 - C_{\alpha f} l_1 = 0$.
It will now be shown that this is also the limit case for Question 2, i.e, in which direction will the following car turn:



Proposition: If $C_{\alpha r} l_2 - C_{\alpha f} l_1 = 0$, then there is exists a steady motion where the car moves with constant velocity without turning.

Answer to Question 2

If the car is not turning, then the side-slip angle is equal for all four tires and the value is given by force equilibrium in the lateral direction:

$$F_y - 2C_{\alpha f}\alpha - 2C_{\alpha r}\alpha = 0$$

The moment of the cornering forces about the center of gravity is in this case equal to

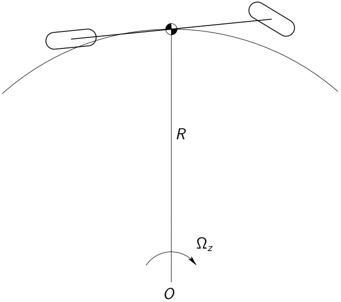
$$M_z = 2\alpha(C_{\alpha r}l_2 - C_{\alpha f}l_1) = 0$$

and the car will continue without turning.

The answer to the Question 2 in the two other cases where $C_{\alpha r}l_2 - C_{\alpha f}l_1 \neq 0$ is the same as for Question 1. The analysis requires more advanced analysis and is beyond the scope of this presentation.

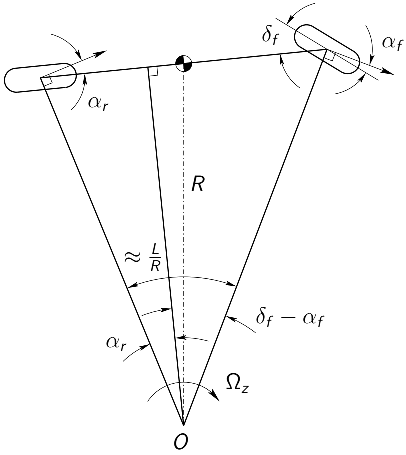
Question 3: Steady state cornering

Question 3: *The car is driving with constant speed on a circle with radius R and origin O. Assume that the speed is increases. In which direction should the steering wheel be turned to make the car stay on the same circle?* The following single-track model will be used in the analysis:



It is assumed that the cornering stiffness of tires are $2C_{\alpha f}$ and $2C_{\alpha r}$, i.e., double the cornering stiffness of the four-wheel model.

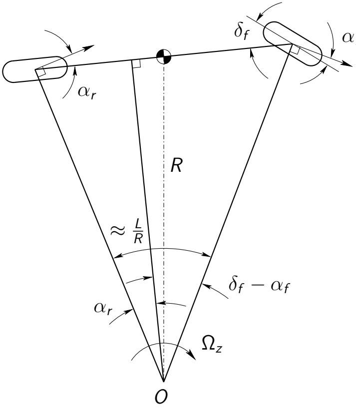
Steady state cornering: Kinematics



Triangle to the left: $\alpha_r + (90^o - \alpha_r) + 90^o = 180^o$
Triangle to the right: $(\delta_f - \alpha_f) + (90^o - (\delta_f - \alpha_f)) + 90^o = 180^o$

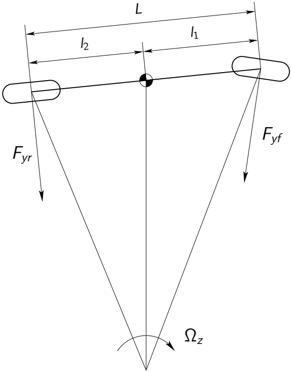
Approximation of the angle at O: $\alpha_r + (\delta_f - \alpha_f) \approx \frac{L}{R}$

Steady state cornering: Kinematics



$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

Steady state cornering



Equations of motion with solutions:

$$F_{yf} + F_{yr} = ma_y = \frac{W}{g}\frac{V^2}{R}$$
$$F_{yf}\, l_1 - F_{yr}\, l_2 = I_z \dot{\Omega}_z = 0$$

$$F_{yf} = ma_y \frac{l_2}{L} = \frac{W}{g}\frac{V^2}{R}\frac{l_2}{L}$$
$$F_{yr} = ma_y \frac{l_1}{L} = \frac{W}{g}\frac{V^2}{R}\frac{l_1}{L}$$

Steady state cornering

Equations of motion with solutions

$$F_{yf} + F_{yr} = ma_y = \frac{W}{g}\frac{V^2}{R}$$
$$F_{yf} l_1 - F_{yr} l_2 = I_z \dot{\Omega}_z = 0$$

$$F_{yf} = ma_y \frac{l_2}{L} = \frac{W}{g}\frac{V^2}{R}\frac{l_2}{L}$$
$$F_{yr} = ma_y \frac{l_1}{L} = \frac{W}{g}\frac{V^2}{R}\frac{l_1}{L}$$

Equations of motions with solutions

$$W_f + W_r = \frac{mg}{2} = \frac{W}{2}$$
$$W_f l_1 - W_r l_2 = I_z \dot{\Omega}_y = 0$$

$$W_f = \frac{mg}{2}\frac{l_2}{L} = \frac{W}{2}\frac{l_2}{L}$$
$$W_r = \frac{mg}{2}\frac{l_1}{L} = \frac{W}{2}\frac{l_1}{L}$$

A comparison of the solutions gives the relations

$$F_{yf} = 2W_f \frac{a_y}{g}, \quad F_{yr} = 2W_r \frac{a_y}{g}$$

Steady state cornering

Using the tire models, $F_{yf} = 2C_{\alpha f}\alpha_f$ and $F_{yr} = 2C_{\alpha r}\alpha_r$, the slip angles can be written as

$$\alpha_f = \frac{F_{yf}}{2C_{\alpha f}} = \frac{W_f}{C_{\alpha f}}\frac{a_y}{g}$$
$$\alpha_r = \frac{F_{yr}}{2C_{\alpha r}} = \frac{W_r}{C_{\alpha r}}\frac{a_y}{g}$$

The steering angle is

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r = \frac{L}{R} + \frac{W_f}{C_{\alpha f}}\frac{a_y}{g} - \frac{W_r}{C_{\alpha r}}\frac{a_y}{g} = \frac{L}{R} + K_{us}\frac{a_y}{g}$$

where the understeer gradient is defined as

$$K_{us} = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}$$

Understeer gradient

$$K_{us} = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} = \frac{W}{2L C_{\alpha f} C_{\alpha r}} (C_{\alpha r} l_2 - C_{\alpha f} l_1)$$

$K_{us} > 0$: The car is said to be understeer.

$K_{us} = 0$: The car is said to be neutral steer.

$K_{us} < 0$: The car is said to be oversteer.

Answer to Question 3

The steering angle is

$$\delta_f = \frac{L}{R} + K_{us}\frac{V^2}{gR} = \frac{L}{R} + K_{us}\frac{a_y}{g}$$

where the understeer gradient can be rewritten as

$$K_{us} = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} = \frac{W}{2LC_{\alpha f}C_{\alpha r}}(C_{\alpha r}l_2 - C_{\alpha f}l_1)$$

The sign of K_{us} gives the answer to Question 3:

- Positive sign: You have to turn the steering wheel clockwise to make the car stay on the same circle.
  The car turns towards the direction of the unperturbed path.
- Equal to zero: You don't have to do anything. Just lie back and relax.
- Negative sign: You have to turn the steering wheel counterclockwise to make the car stay on the same circle.

Understeer gradient K_{us}

Consider the relation

$$\delta_f = \frac{L}{R} + \underbrace{\left(\frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}\right)}_{=K_{us}} \frac{V^2}{gR}$$

Interpretation:

The understeer gradient is equal to the difference between the ratios of the load and cornering stiffness at the front and rear wheels, respectively.

Understeer gradient K_{us}

The understeer gradient can be rewritten as

$$\delta_f = \frac{L}{R} + \underbrace{\frac{W}{2LC_{\alpha f}C_{\alpha r}}(C_{\alpha r}l_2 - C_{\alpha f}l_1)}_{=K_{us}}\frac{V^2}{gR}$$

Interpretation:

The sign of the understeer gradient depends on the difference between the products of the cornering stiffness and the distance to the center of gravity at the rear and front wheels, respectively.

Understeer gradient K_{us}

Why does the sign of $C_{\alpha r} l_2 - C_{\alpha f} l_1$ give the answer question 3?

To give a more direct interpretation we shall only consider the equilibrium of moments

$$l_1 F_{yf} - l_2 F_{yr} = 2 l_1 C_{\alpha f} \alpha_f - 2 l_2 C_{\alpha r} \alpha_r = 0,$$

which gives the relation

$$l_1 C_{\alpha f} \alpha_f = l_2 C_{\alpha r} \alpha_r,$$

and the relation

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

between the angles.

The case: Neutral steer

Assume that the radius of the curve R is constant and the speed V increases. In this case $l_1 C_{\alpha f} - l_2 C_{\alpha r} = 0$ and the relation

$$l_1 C_{\alpha f} \alpha_f = l_2 C_{\alpha r} \alpha_r$$

gives $\alpha_f = \alpha_r$ and

$$\delta_f = \frac{L}{R} + \underbrace{\alpha_f - \alpha_r}_{=0}$$

does not depend on the speed.

The case: Understeer

In this case $l_2 C_{\alpha r} - l_1 C_{\alpha f} > 0$ and it follows from

$$l_1 C_{\alpha f} \alpha_f = l_2 C_{\alpha r} \alpha_r$$

that the increase of α_f has to larger than the increase of α_r, when V increases.

Hence, the steering angle

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

has to be increased.

The case: Oversteer

In this case $l_2 C_{\alpha r} - l_1 C_{\alpha f} < 0$ and it follows from

$$l_1 C_{\alpha f} \alpha_f = l_2 C_{\alpha r} \alpha_r$$

that the increase of α_r has to larger than the increase of α_f, when V increases.

Hence, the steering angle

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

has to be increased.
Observation: If $K_{us} < 0$ and $V = V_{crit} = \sqrt{gL/-K_{us}}$ then

$$\delta_f = \frac{L}{R} + K_{us} \frac{V^2}{gR} = 0$$

and does not depend on R.

Handling at low speeds

If the velocity of the car is small, then

$$\delta_f \approx \frac{L}{R}$$

Furthermore,

$$V = R\dot{\theta}$$

where θ is the direction of the car.

A simple kinematic model of the car:

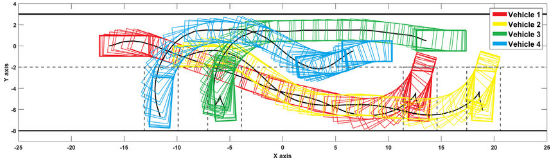
$$\dot{x} = V\cos\theta$$
$$\dot{y} = V\sin\theta$$
$$\dot{\theta} = \frac{V\delta_f}{L}$$

A kinematic model

Adding longitudinal dynamics gives the following model:

$$\begin{aligned} \dot{x} &= V\cos\theta \\ \dot{y} &= V\sin\theta \\ \dot{\theta} &= \frac{V\delta_f}{L} \\ m\dot{V} &= F \end{aligned}$$

A typical application is motion planning at, e.g., a parking lot:



New course next year:
TSFS12 Autonomous Vehicles - Planning, Control, and Learning Systems

Split μ

In some cases there is a difference between the coefficient of friction on the right- and left-hand side

Split μ

Assume that the car i back-wheel driven and the coefficient of friction is μ_r to the right μ_l to the left.

Assume that the normal force is equal on the left and right rear wheel, i.e., $W_r/2$.

Maximum acceleration is obtained when

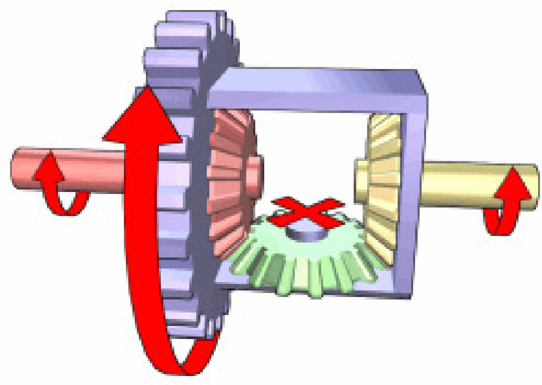
$$F_r = \mu_h W_r/2 + \mu_v W_r/2 = \frac{\mu_h + \mu_v}{2} W_r$$

How can we distribute the force between the right and left wheel to reach maximum acceleration

In the case of the car is braking, the ABS system will take care of the distribution of braking forces.

If the car is accelerating, then an active differential can solve the problem.

Differential gear

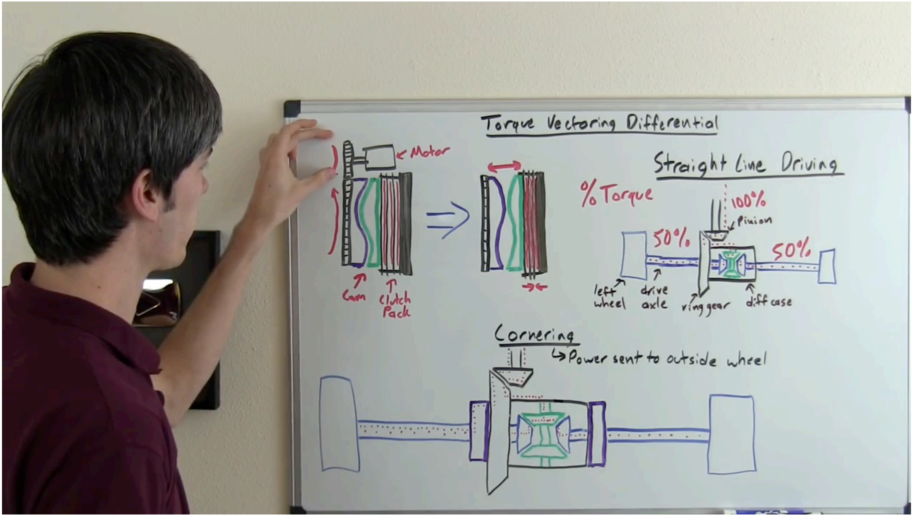


Differential gear

Purpose

- The wheels can rotate with different angular speed in a curve.
- The torque is distributed equally between the wheels.

With an active differential it is possible to partially lock the differential an distribute the torque between the wheels.


Torque Vectoring Differential
Motor
Cam
Clutch Pack
Straight Line Driving
% Torque
100%
Pinion
50%
50%
left wheel
drive axle
ring gear
diff case
Cornering
↳ Power sent to outside wheel

Steering geometry: Ackermann

Figure 5.2 shows the relationship between the steering angles δ_o and δ_i in case of pure rolling without lateral slip at low speed.

With $d = |OF|$ we get the relations

$$\cot \delta_o = \frac{d + B}{L}$$

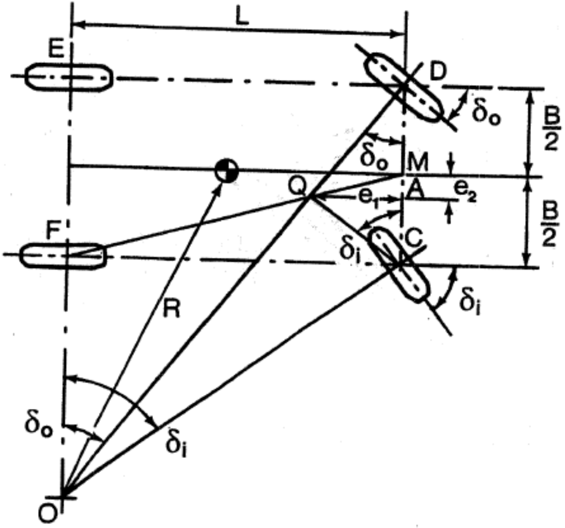
$$\cot \delta_i = \frac{d}{L}$$

and

$$\cot \delta_o - \cot \delta_i = \frac{B}{L}$$

Steering geometry: Ackermann

Figure 5.2



Steering geometry: Ackermann

Two other relations can be seen in the figure

$$\cot \delta_o = \frac{B/2 + e_2}{e_1}$$

$$\cot \delta_i = \frac{B/2 - e_2}{e_1}$$

and

$$\cot \delta_o - \cot \delta_i = \frac{2e_2}{e_1}$$

it follows from the relations above that

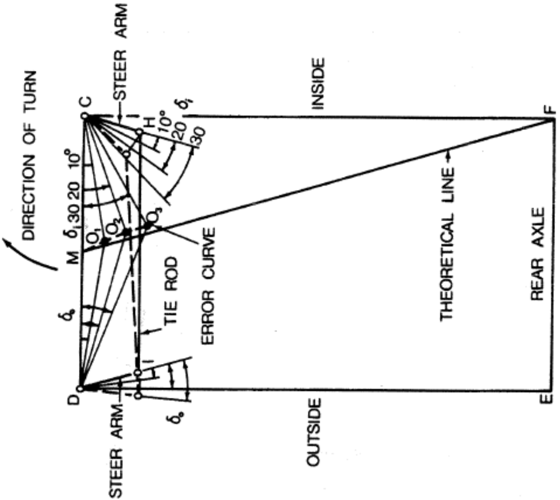
$$\frac{e_2}{e_1} = \frac{B/2}{L}$$

Hence, the point Q is located on the line MF.

Steering geometry: Ackermann

Figure 5.4 shows an example of a steering linkage and how the points O_1, O_2 och O_3, corresponding to the point Q, are located in relation to the line MF.

Figure 5.4


DIRECTION OF TURN
STEER ARM
C
H
10°
20
30
δ_i
INSIDE
F
M
O_1
O_2
O_3
δ_i 30 20 10°
δ_o
TIE ROD
ERROR CURVE
THEORETICAL LINE
REAR AXLE
D
E
STEER ARM
δ_o
OUTSIDE