Vehicle Dynamics and Control

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Lecture 6

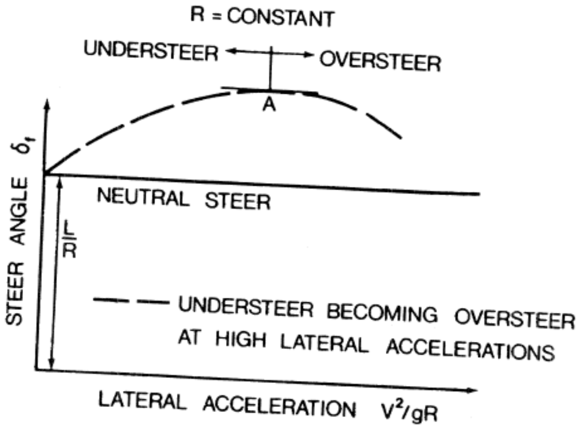
Kurvtagning

I kapitel 5.4 studeras kurvtagning för tre fall:

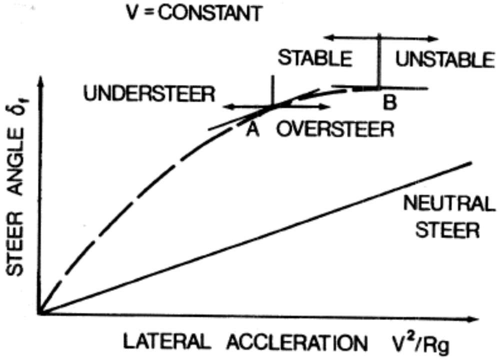
- Konstant radie, figur 5.15.
- Konstant hastighet, figur 5.16.
- Konstant styrvinkel, figur 5.17.

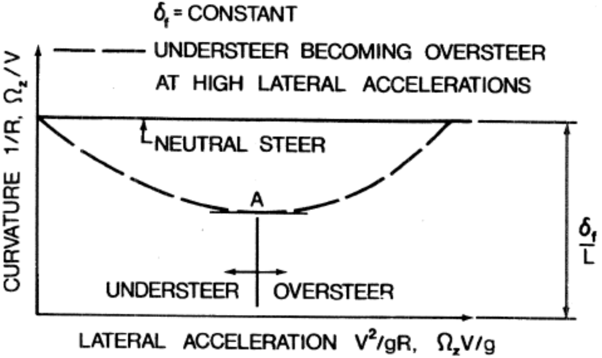
Jag kommer nu att gå igenom sambandet mellan kurvorna i figurerna och däckens (olinjära) egenskaper.

Kurvtagning: Figur 5.15



Kurvtagning: Figur 5.16

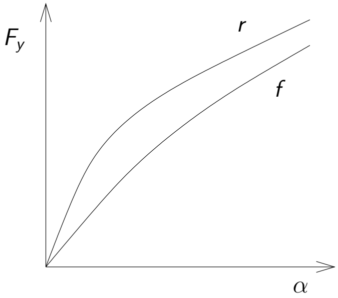




δ_f = CONSTANT
UNDERSTEER BECOMING OVERSTEER AT HIGH LATERAL ACCELERATIONS
NEUTRAL STEER
A
UNDERSTEER
OVERSTEER
CURVATURE 1/R, Ω_z/V
LATERAL ACCELERATION V^2/gR, $\Omega_z V/g$
$\frac{\delta_f}{L}$

Kurvtagning: Utgår från en olinjär modell

Utgår från att F_{yf} och F_{yr} är givna som funktioner av α.



Normalkrafter och laterala krafter

De laterala krafterna ges av

$$F_{yf} + F_{yr} = ma_y$$
$$F_{yf} l_1 - F_{yr} l_2 = 0$$

Normalkrafterna ges av

$$F_{zf} + F_{zr} = mg$$
$$F_{zf} l_1 - F_{zr} l_2 = 0$$

Vi kan som tidigare beräkna krafterna, men genom att studera ekvationerna kan man inse att:

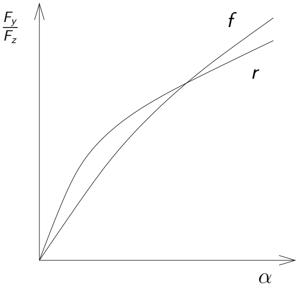
$$\frac{F_{yf}}{F_{zf}} = \frac{F_{yr}}{F_{zr}} = \frac{a_y}{g} = \frac{V^2}{gR}$$

Arbetsgång

Vi beräknar först normalkrafterna

$$F_{zf} = mg\frac{l_2}{L}$$
$$F_{zr} = mg\frac{l_1}{L}$$

Därefter ritar vi de skalade kurvorna F_{yf}/F_{zf} och F_{yr}/F_{zr}. För en baktung bil får vi kurvorna

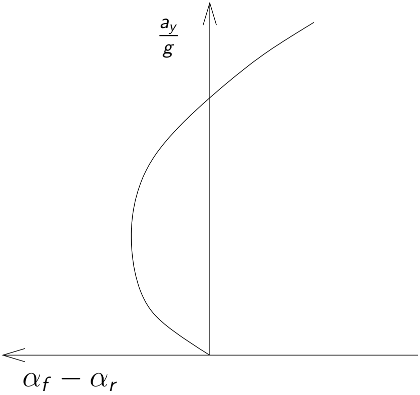


Arbetsgång

För en given kurvradie och hastighet är som bekant

$$\frac{F_{yf}}{F_{zf}} = \frac{F_{yr}}{F_{zr}} = \frac{a_y}{g} = \frac{V^2}{gR}$$

Genom att ta skillnaden mellan vinklarna α_f och α_r i horisontell led får vi kurvan

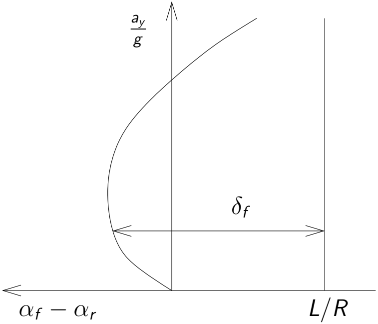


Konstant radie

Studerar nu fallet att radien R är konstant. Styrvinkeln ges som vanligt av ekvationen

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

Detta samband kan avläsas i figuren



Jämför med figur 5.15.

Understyrningsgradienten K_{us}

I det olinjära fallet med konstant kurvradie ges understyrningsgradienten av derivatan

$$K_{us} = \frac{d(\delta_f)}{d(a_y/g)}$$

Konsistent med definitionen i det linjära fallet då

$$\delta_f = \frac{L}{R} + K_{us}\frac{a_y}{g}$$

För ökande hastighet så gäller som tidigare att

- δ_f ökar om $K_{us} > 0$
- δ_f oförändrad om $K_{us} = 0$
- δ_f minskar om $K_{us} < 0$

Konstant hastighet

Studerar nu fallet att hastigheten V är konstant.

Det gäller fortfarande att

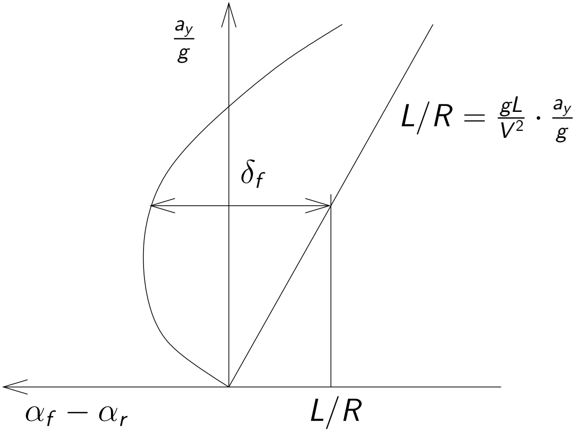
$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

där

$$\frac{L}{R} = \frac{gL}{V^2} \cdot \frac{V^2}{gR} = \underbrace{\frac{gL}{V^2}}_{konst.} \cdot \frac{a_y}{g}$$

Konstant hastighet

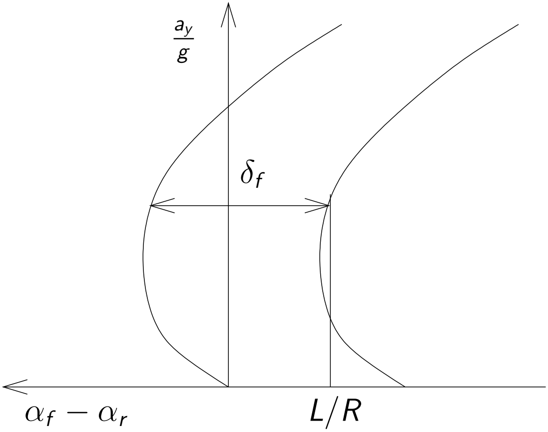
Sambandet mellan styrvinkeln δ_f och laterala accelerationen kan läsas av i figuren



Jämför med figur 5.16.

Konstant styrvinkel

Slutligen studerar vi fallet där styrvinkeln δ_f är konstant.



I detta fall kan vi avläsa hur krökningen $1/R$ beror av a_y/g.

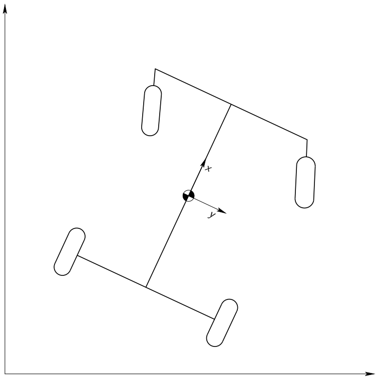
Jämför med figur 5.17.

Stability of a shopping cart

The Americana's CC-12 Double Basket Convenience Metal Wire Shopping Cart (Available in ten different colors):

Stability of a shopping cart:

The stability of the shopping cart will now be analyzed. Assume that shopping cart is moving on a straight line and at some point the steady motion is perturbed. We begin with the derivation of the equations of motion and a body-fixed coordinate system attached to the center of mass of the cart will be used in the analysis:



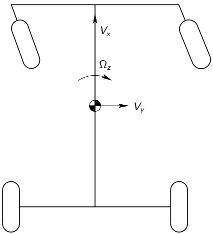
Stability of a shopping cart: Body-fixed coordinate system

For any time dependant vector **u** measured in a body-fixed coordinate system rotating with angular velocity ω, the absolute rate of change is

$$\frac{d\mathbf{u}}{dt} = \left.\frac{d\mathbf{u}}{dt}\right|_{\text{rel}} + \omega \times \mathbf{u}$$

where the first term in the right-hand side denotes the rate of change of the vector seen in the body-fixed coordinate system.

Stability of a shopping cart: Body-fixed coordinate system



With the velocity vector as $\mathbf{u}$ and the system rotating around the z-axis, (pointing downwards) with angular velocity Ω_z, the relation

$$\frac{d\mathbf{u}}{dt} = \left.\frac{d\mathbf{u}}{dt}\right|_{\text{rel}} + \omega \times \mathbf{u}$$

becomes

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \dot{V}_x \\ \dot{V}_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_z \end{pmatrix} \times \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \dot{V}_x - \Omega_z V_y \\ \dot{V}_y + \Omega_z V_x \\ 0 \end{pmatrix}$$

Transient beteende

Använder ett koordinatsystem som är fixt i förhållande till bilen.

Första ordningens approximation av hastighetsändringen i x-led:

$$(V_x + \Delta V_x)\cos\Delta\theta - (V_y + \Delta V_y)\sin\Delta\theta - V_x \sim \Delta V_x - V_y\Delta\theta$$

Dela med Δt och låt Δt gå mot noll:

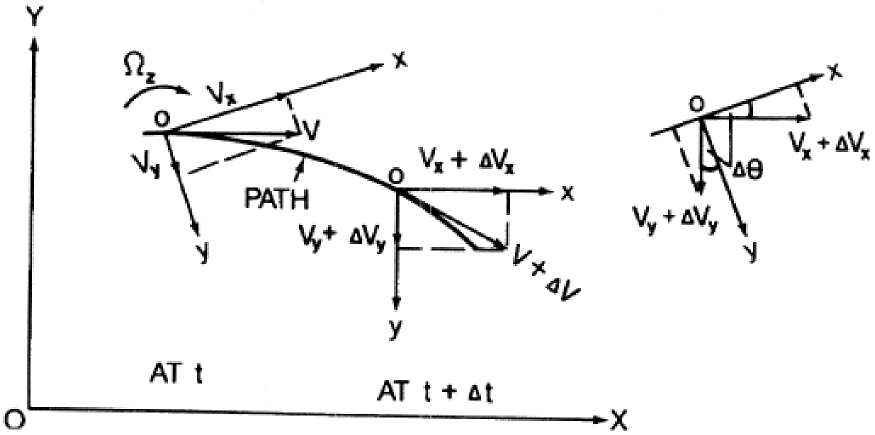
$$a_x = \frac{dV_x}{dt} - V_y\frac{d\theta}{dt} = \dot{V}_x - V_y\Omega_z$$

På samma sätt fås i y-led:

$$(V_y + \Delta V_y)\cos\Delta\theta + (V_x + \Delta V_x)\sin\Delta\theta - V_y \sim \Delta V_y + V_x\Delta\theta$$

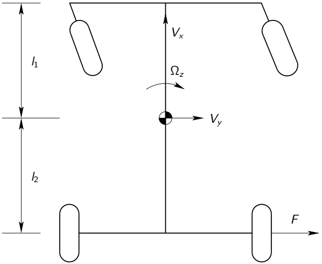
$$a_y = \frac{dV_y}{dt} + V_x\frac{d\theta}{dt} = \dot{V}_y + V_x\Omega_z$$

Figur 5.18



Y
X
O
Ω_z
V_x
x
o
V
V_y
y
PATH
$V_x + \Delta V_x$
$V_y + \Delta V_y$
$V + \Delta V$
AT t
AT t + Δt
Δθ

Stability of a shopping cart



With $a_x = \dot{V}_x - \Omega_z V_y$ and $a_y = \dot{V}_y + \Omega_z V_x$ the plane equations of motion for the shopping cart become

$$m(\dot{V}_x - \Omega_z V_y) = 0$$
$$m(\dot{V}_y + \Omega_z V_x) = F$$
$$I_z \dot{\Omega}_z = -l_2 F$$

It is assumed that the side velocity at the rear is equal to zero, i.e.,

$$V_y - l_2 \Omega_z = 0$$

Stability of a shopping cart

The three equations

$$
\begin{aligned}
I_z \dot{\Omega}_z &= -l_2 F \\
m(\dot{V}_y + \Omega_z V_x) &= F \\
V_y &= l_2 \Omega_z
\end{aligned}
$$

give

$$I_z \dot{\Omega}_z = -l_2 F = -l_2 m(\dot{V}_y + \Omega_z V_x) = -l_2 m(l_2 \dot{\Omega}_y + \Omega_z V_x)$$

which can be rewritten as

$$(I_z + m l_2^2)\dot{\Omega}_z + m l_2 V_x \Omega_z = 0$$

Stability of a shopping cart

The solution of the differential equation $(I_z + ml_2^2)\dot{\Omega}_z + ml_2 V_x \Omega_z = 0$ is

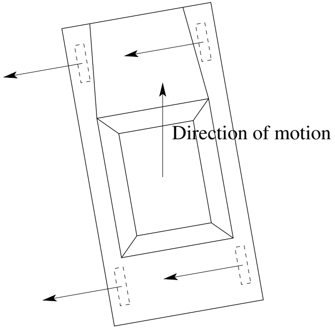
$$\Omega_z = e^{-t/\tau}, \quad \tau = \frac{I_z + ml_2^2}{ml_2 V_x}$$

It can be seen that the yaw rate Ω_z tends to zero as time tends to infinity, if $V_x > 0$. However, this does not imply that the direction of the cart will return to the direction the cart was traveling in before the perturbation.

If the basic motion is defined as $\Omega_z = 0$ and $V_y = 0$, then the perturbed motion approaches the basic motion and the system is stable.

Above it was assumed that the cart was traveling forward ($V_x > 0$). If the cart is pushed backwards ($V_x < 0$) the cart becomes unstable ($\tau < 0$).

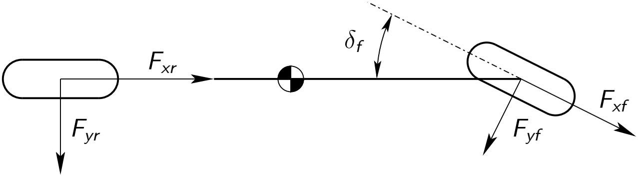
Now, we shall study the stability of the car after a disturbance.



We have seen in the previous lecture that the car will tend to turn in different directions depending on the sign of the understeer gradient K_{us}. Today's question is whether the car will tend to return to the basic motion, that we define in the same way as before to be $\Omega_z = 0$ and $V_y = 0$.

Stability: Equations of motion

The forces acting on the vehicle:



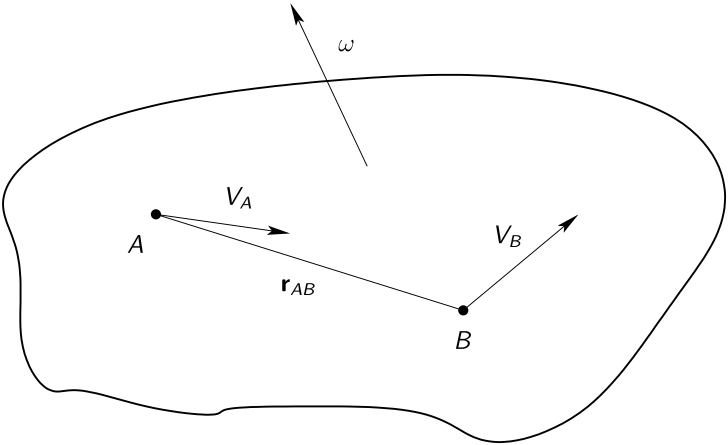
Equations of motions:

$$
\begin{aligned}
m(\dot{V}_x - V_y\Omega_z) &= F_{xf}\cos\delta_f + F_{xr} - F_{yf}\sin\delta_f \\
m(\dot{V}_y + V_x\Omega_z) &= F_{yr} + F_{yf}\cos\delta_f + F_{xf}\sin\delta_f \\
I_z\dot{\Omega}_z &= l_1 F_{yf}\cos\delta_f - l_2 F_{yr} + l_1 F_{xf}\sin\delta_f
\end{aligned}
$$

The left-hand side is the same as for the shopping cart and V_x and V_y are the velocities in the body-fixed coordinate system.

Stability: Kinematics

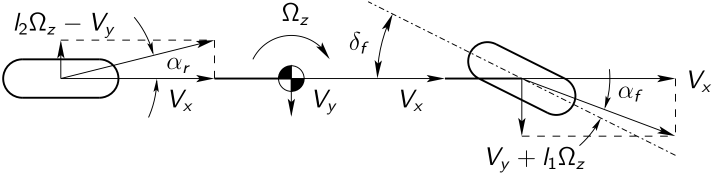
In general:



$$\mathbf{v}_B = \mathbf{v}_A + \omega \times \mathbf{r}_{AB}$$

Stability: Kinematics

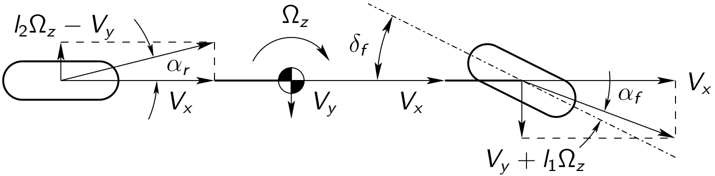
With A at center of gravity and B at the front/rear wheel:



$$\text{Front wheel:} \quad \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_z \end{pmatrix} \times \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} V_x \\ V_y + l_1\Omega_z \\ 0 \end{pmatrix}$$

$$\text{Rear wheel:} \quad \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_z \end{pmatrix} \times \begin{pmatrix} -l_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} V_x \\ V_y - l_2\Omega_z \\ 0 \end{pmatrix}$$

Stability: Kinematics



Kinematics from the figure

$$\alpha_f = \delta_f - \frac{l_1\Omega_z + V_y}{V_x}$$

$$\alpha_r = \frac{l_2\Omega_z - V_y}{V_x}$$

Model for tire forces

$$F_{yf} = 2C_{\alpha f}\alpha_f$$

$$F_{yr} = 2C_{\alpha r}\alpha_r$$

From now it will be assumed that the longitudinal velocity V_x is constant and will play the role of a parameter in the analysis.

Stability

From the previous slides:

$$
\begin{aligned}
m(\dot{V}_y + V_x\Omega_z) &= F_{yr} + F_{yf}\cos\delta_f + F_{xf}\sin\delta_f \\
I_z\dot{\Omega}_z &= l_1 F_{yf}\cos\delta_f - l_2 F_{yr} + l_1 F_{xf}\sin\delta_f
\end{aligned}
$$

$$
F_{yf} = 2C_{\alpha f}\left(\delta_f - \frac{l_1\Omega_z + V_y}{V_x}\right), \quad F_{yr} = 2C_{\alpha r}\left(\frac{l_2\Omega_z - V_y}{V_x}\right)
$$

Can be rearranged in the form

$$
m\dot{V}_y + \Bigg[\underbrace{\frac{2C_{\alpha f} + 2C_{\alpha r}}{V_x}}_{a_1}\Bigg]V_y + \Bigg[\underbrace{mV_x + \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x}}_{a_2}\Bigg]\Omega_z = 2C_{\alpha f}\delta_f(t)
$$

$$
I_z\dot{\Omega}_z + \Bigg[\underbrace{\frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x}}_{a3}\Bigg]V_y + \Bigg[\underbrace{\frac{2l_1^2 C_{\alpha f} + 2l_2^2 C_{\alpha r}}{V_x}}_{a_4}\Bigg]\Omega_z = 2l_1 C_{\alpha f}\delta_f(t)
$$

Stability

The system

$$m\dot{V}_y + \underbrace{\left[\frac{2C_{\alpha f} + 2C_{\alpha r}}{V_x}\right]}_{a_1} V_y + \underbrace{\left[mV_x + \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x}\right]}_{a_2} \Omega_z = 2C_{\alpha f}\delta_f(t)$$

$$I_z\dot{\Omega}_z + \underbrace{\left[\frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x}\right]}_{a3} V_y + \underbrace{\left[\frac{2l_1^2 C_{\alpha f} + 2l_2^2 C_{\alpha r}}{V_x}\right]}_{a_4} \Omega_z = 2l_1 C_{\alpha f}\delta_f(t)$$

can be written in matrix form

$$M\dot{\mathbf{x}} + A\mathbf{x} = \mathbf{u}(t)$$

where

$$M = \begin{bmatrix} m & 0 \\ 0 & I_z \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 2C_{\alpha f}\delta_f(t) \\ 2l_1 C_{\alpha f}\delta_f(t) \end{bmatrix}$$

Stability

From the basic course in linear algebra we now that the solution of a system of linear equations

$$\dot{\mathbf{x}} = A\mathbf{x}$$

is a linear combination of vectors in the form

$$e^{\lambda t} X$$

where λ is an eigenvalue of A and X is the corresponding eigenvector. Assume that the solution of $M\dot{\mathbf{x}} + A\mathbf{x} = 0$ is of the form

$$\mathbf{x}(t) = e^{st} X$$

where s i a constant, possibly complex a number. Substitute the expression into the system of differential equations:

$$M\dot{\mathbf{x}} + A\mathbf{x} = \Big/ \dot{\mathbf{x}} = \frac{d}{dt} e^{st} X = s e^{st} X \Big/ = e^{st}(sM + A)X = 0$$

Stability

The relation

$$e^{st}(sM + A)X = 0$$

has to fulfilled for all values of t and the only possibility is that the vector X is the solution of the homogeneous system of equations

$$(sM + A)X = 0$$

A non-trivial solution ($X \neq 0$) exists if and only

$$det(sM + A) = 0$$

Stability

To summarize the discussion. The procedure to calculate the solutions of the form

$$\mathbf{x}(t) = e^{st} X$$

of the system of differential equations

$$M\dot{\mathbf{x}} + A\mathbf{x} = 0$$

is to first determine the eigenvalues s by solving the characteristic equation

$$det(sM + A) = 0$$

and then for each eigenvalue s determine the corresponding eigenvectors by solving system of linear equations

$$(sM + A)X = 0$$

Stability

With

$$M = \begin{bmatrix} m & 0 \\ 0 & I_z \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

where a_1, a_2, a_3, and a_4 was defined by

$$m\dot{V}_y + \Bigg[\underbrace{\frac{2C_{\alpha f} + 2C_{\alpha r}}{V_x}}_{a_1}\Bigg] V_y + \Bigg[\underbrace{mV_x + \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x}}_{a_2}\Bigg] \Omega_z = 2C_{\alpha f}\delta_f(t)$$

$$I_z\dot{\Omega}_z + \Bigg[\underbrace{\frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x}}_{a3}\Bigg] V_y + \Bigg[\underbrace{\frac{2l_1^2 C_{\alpha f} + 2l_2^2 C_{\alpha r}}{V_x}}_{a_4}\Bigg] \Omega_z = 2l_1 C_{\alpha f}\delta_f(t)$$

the characteristic equation becomes

$$\begin{aligned} \det(sM + A) &= \det \begin{bmatrix} sm + a_1 & a_2 \\ a_3 & sI_z a_4 \end{bmatrix} \\ &= mI_z s^2 + (I_z a_1 + m a_4)s + (a_1 a_4 - a_2 a_3) = 0 \end{aligned}$$

Stability

The system is stable if and only if the solutions of

$$\det(sM + A) = mI_z s^2 + (I_z a_1 + ma_4)s + (a_1 a_4 - a_2 a_3) = 0$$

are all in the left half of the complex plane. The solutions of a second order equation

$$s^2 + ps + q = 0$$

is given by the formula

$$s = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

I can be seen that solutions are all in the left half of the complex plane if and only if $p > 0$ and $q > 0$. It easy to verify that $mI_z > 0$, and $I_z a_1 + ma_4 > 0$. Hence, stability depends only on the sign of the expression $a_1 a_4 - a_2 a_3$.

Stability

It can be shown that

$$a_1 a_4 - a_2 a_3 > 0$$

where a_1, a_2, a_3, and a_4 was defined by

$$m\dot{V}_y + \underbrace{\left[\frac{2C_{\alpha f} + 2C_{\alpha r}}{V_x}\right]}_{a_1} V_y + \underbrace{\left[mV_x + \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x}\right]}_{a_2} \Omega_z = 2C_{\alpha f}\delta_f(t)$$

$$I_z\dot{\Omega}_z + \underbrace{\left[\frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x}\right]}_{a3} V_y + \underbrace{\left[\frac{2l_1^2 C_{\alpha f} + 2l_2^2 C_{\alpha r}}{V_x}\right]}_{a_4} \Omega_z = 2l_1 C_{\alpha f}\delta_f(t)$$

is equivalent to the condition

$$L + \frac{V_x^2}{g}\left(\frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}\right) = L + \frac{V_x^2}{g} K_{us} > 0$$

Stability: Answer to the Question of the day

The condition

$$L + \frac{V_x^2}{g}\left(\frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}\right) = L + \frac{V_x^2}{g} K_{us} > 0$$

can only be violated if

$$K_{us} < 0$$

i.e. the car is understeer and

$$V \geq \sqrt{\frac{gL}{-K_{us}}} = V_{crit}$$

Answer to the question:

The car is unstable if and only if $K_{us} < 0$ and $V \geq V_{crit}$.

Dragbil med semi-trailer: Kurvtagning

Figur 5.26 visar en förenklad modell.

Styrvinkel

$$\delta_f = \frac{L_t}{R} + \underbrace{\left(\frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}\right)}_{K_{us,t}} \frac{V^2}{gR}$$

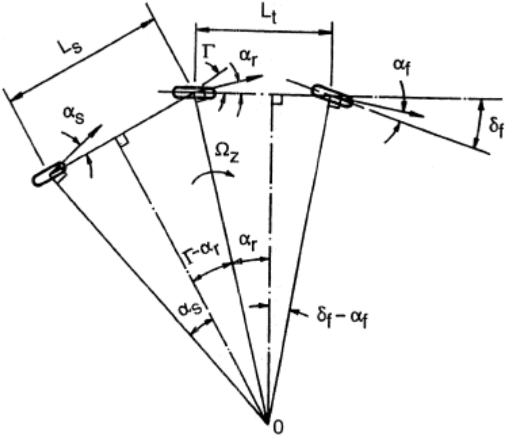
Vinkeln mellan dragbil och semi-trailer

$$\Gamma = \frac{L_s}{R} + \underbrace{\left(\frac{W_r}{C_{\alpha r}} - \frac{W_s}{C_{\alpha s}}\right)}_{K_{us,s}} \frac{V^2}{gR}$$

Förstärkning:

$$\frac{\Gamma}{\delta_f} = \frac{L_s/R + K_{us,s}(V^2/gR)}{L_t/R + K_{us,t}(V^2/gR)}$$

Kurvtagning: Figur 5.26



Dragbil med semi-trailer: Fall 1

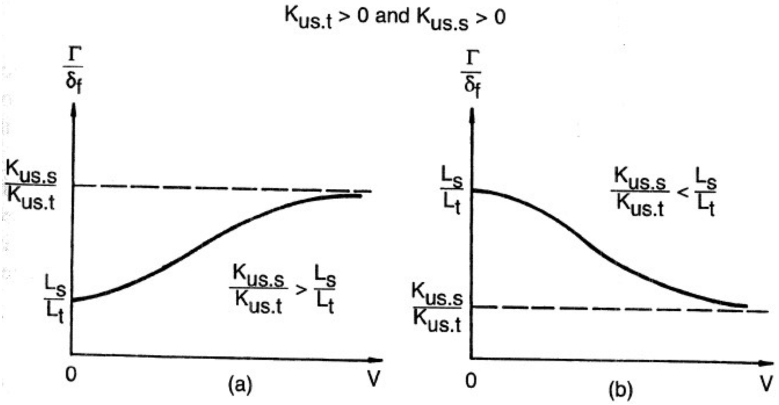
$K_{us,t}$ och $K_{us,s}$ är båda positiva.

Förstärkningen Γ/δ_f är alltid postiv.

Förstärkningen är växande om $K_{us,s}/K_{us,t} > L_s/L_t$ och avtagande om $K_{us,s}/K_{us,t} < L_s/L_t$.

Se figur 5.27.

Kurvtagning: Figur 5.27



Dragbil med semi-trailer: Fall 2

$K_{us,t}$ är positiv och $K_{us,s}$ är negativ.

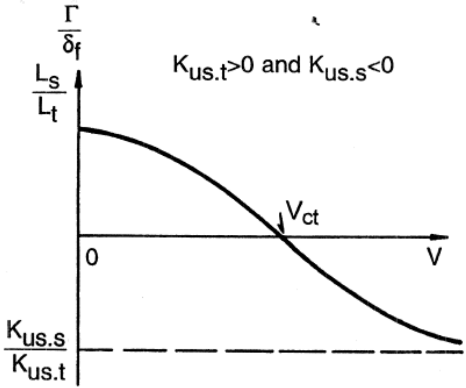
För hastigheter över

$$V_{ct} = \sqrt{\frac{gL_s}{-K_{us,s}}}$$

är förstärkningen negativ.

Se figur 5.28.

Kurvtagning: Figur 5.28



Dragbil med semi-trailer: Fall 3

$K_{us,t}$ är negativ och $K_{us,s}$ är positiv.

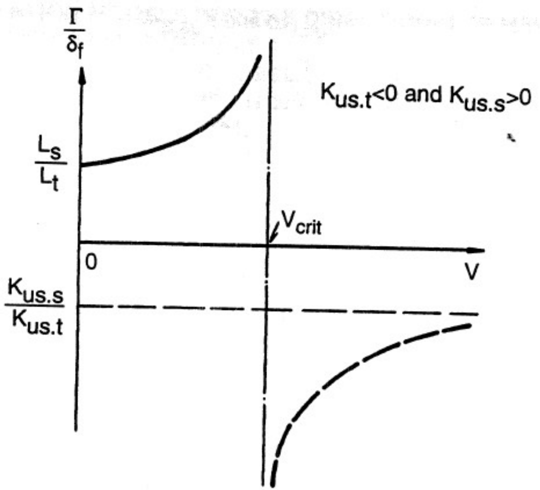
När hastigheten närmar sig det kritiska värdet

$$V_{crit} = \sqrt{\frac{gL_t}{-K_{us,t}}}$$

går förstärkningen mot oändligheten.

Bilen fälls ihop som en fällkniv ("Jackknifing"). Se figur 5.29.

Kurvtagning: Figur 5.29



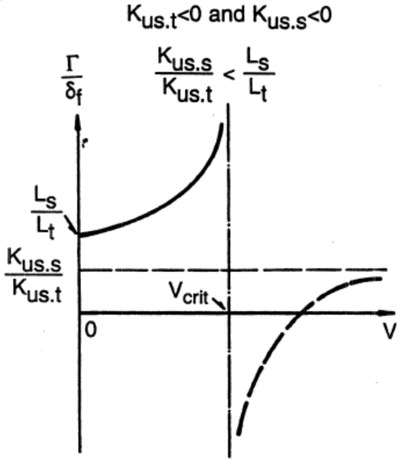
Dragbil med semi-trailer: Fall 4

$K_{us,t}$ och $K_{us,s}$ är båda negativa och $K_{us,s}/K_{us,t} < L_s/L_t$.

Det händer samma sak som i föregående fall.

Se figur 5.30.

Kurvtagning: Figur 5.30



Dragbil med semi-trailer: Fall 5

$K_{us,t}$ och $K_{us,s}$ är båda negativa och $K_{us,s}/K_{us,t} > L_s/L_t$.

Förstärkningen blir negativ för hastigheter över den karakteristiska hastigheten

$$V_{ct} = \sqrt{\frac{gL_s}{-K_{us,s}}}$$

När hastigheten närmar sig den kritiska hastigheten

$$V_{crit} = \sqrt{\frac{gL_t}{-K_{us,t}}}$$

går kvoten mot $-\infty$ och trailern svänger ut ("trailer swing").

Se figur 5.31.

Kurvtagning: Figur 5.31