

## Vehicle Dynamics and Control

Jan Åslund  
 jan.aslund@liu.se  
 Associate Professor

Dept. Electrical Engineering  
 Vehicular Systems  
 Linköping University  
 Sweden

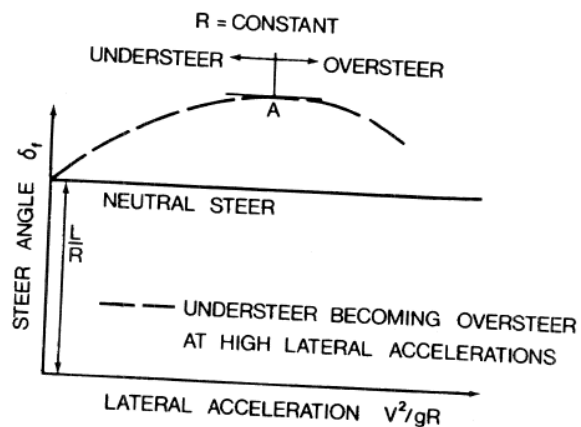
### Lecture 6

I kapitel 5.4 studeras kurvtagning för tre fall:

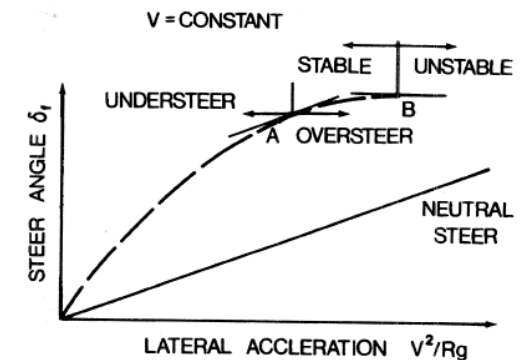
- Konstant radie, figur 5.15.
- Konstant hastighet, figur 5.16.
- Konstant styrvinkel, figur 5.17.

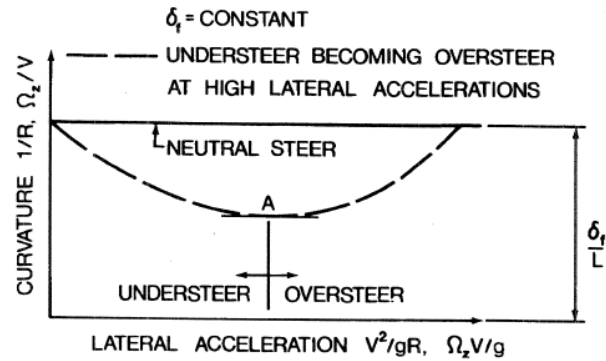
Jag kommer nu att gå igenom sambandet mellan kurvorna i figurerna och däckens (olinjära) egenskaper.

## Kurvtagning: Figur 5.15



## Kurvtagning: Figur 5.16





## Normalkrafter och laterala krafter

De laterala krafterna ges av

$$\begin{aligned} F_{yf} + F_{yr} &= ma_y \\ F_{yf}l_1 - F_{yr}l_2 &= 0 \end{aligned}$$

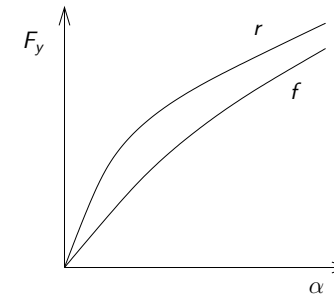
Normalkrafterna ges av

$$\begin{aligned} F_{zf} + F_{zr} &= mg \\ F_{zf}l_1 - F_{zr}l_2 &= 0 \end{aligned}$$

Vi kan som tidigare beräkna krafterna, men genom att studera ekvationerna kan man inse att:

$$\frac{F_{yf}}{F_{zf}} = \frac{F_{yr}}{F_{zr}} = \frac{a_y}{g} = \frac{V^2}{gR}$$

Utgår från att  $F_{yf}$  och  $F_{yr}$  är givna som funktioner av  $\alpha$ .

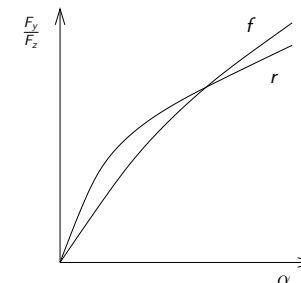


## Arbetsgång

Vi beräknar först normalkrafterna

$$\begin{aligned} F_{zf} &= mg \frac{l_2}{L} \\ F_{zr} &= mg \frac{l_1}{L} \end{aligned}$$

Därefter ritar vi de skalade kurvorna  $F_{yf}/F_{zf}$  och  $F_{yr}/F_{zr}$ . För en baktung bil får vi kurvorna

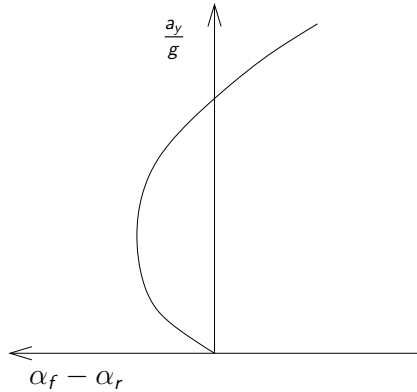


## Arbetsgång

För en given kurvradie och hastighet är som bekant

$$\frac{F_{yf}}{F_{zf}} = \frac{F_{yr}}{F_{zr}} = \frac{a_y}{g} = \frac{V^2}{gR}$$

Genom att ta skillnaden mellan vinklarna  $\alpha_f$  och  $\alpha_r$  i horisontell led får vi kurvan

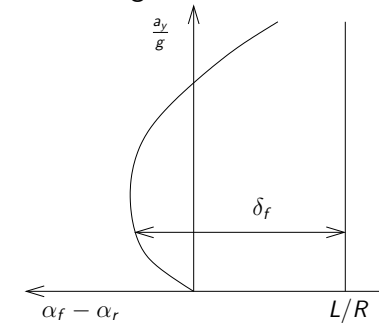


## Konstant radie

Studerar nu fallet att radien  $R$  är konstant. Styrvinkeln ges som vanligt av ekvationen

$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

Detta samband kan avläsas i figuren



Jämför med figur 5.15.

## Understyrningsgradienten $K_{us}$

I det olinjära fallet med konstant kurvradie ges understyrningsgradienten av derivatan

$$K_{us} = \frac{d(\delta_f)}{d(a_y/g)}$$

Konsistent med definitionen i det linjära fallet då

$$\delta_f = \frac{L}{R} + K_{us} \frac{a_y}{g}$$

För ökande hastighet så gäller som tidigare att

- $\delta_f$  ökar om  $K_{us} > 0$
- $\delta_f$  oförändrad om  $K_{us} = 0$
- $\delta_f$  minskar om  $K_{us} < 0$

## Konstant hastighet

Studerar nu fallet att hastigheten  $V$  är konstant.

Det gäller fortfarande att

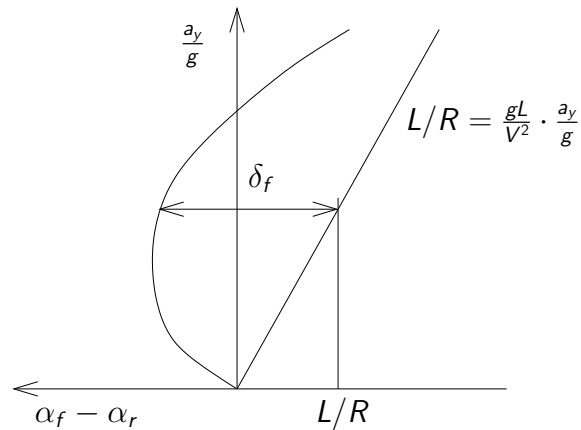
$$\delta_f = \frac{L}{R} + \alpha_f - \alpha_r$$

där

$$\frac{L}{R} = \frac{gL}{V^2} \cdot \frac{V^2}{gR} = \underbrace{\frac{gL}{V^2}}_{\text{konst.}} \cdot \frac{a_y}{g}$$

## Konstant hastighet

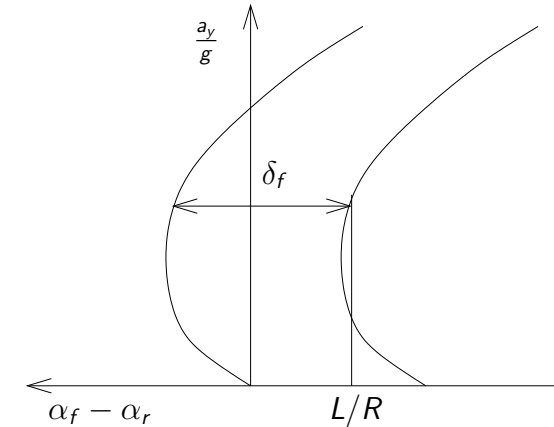
Sambandet mellan styrvinkeln  $\delta_f$  och laterala accelerationen kan läsas av i figuren



Jämför med figur 5.16.

## Konstant styrvinkel

Slutligen studerar vi fallet där styrvinkeln  $\delta_f$  är konstant.



I detta fall kan vi avläsa hur krökningen  $1/R$  beror av  $a_y/g$ .

Jämför med figur 5.17.

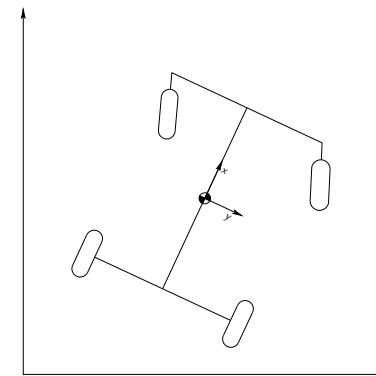
## Stability of a shopping cart

The Americana's CC-12 Double Basket Convenience Metal Wire Shopping Cart (Available in ten different colors):



## Stability of a shopping cart:

The stability of the shopping cart will now be analyzed. Assume that shopping cart is moving on a straight line and at some point the steady motion is perturbed. We begin with the derivation of the equations of motion and a body-fixed coordinate system attached to the center of mass of the cart will be used in the analysis:



# Stability of a shopping cart: Body-fixed coordinate system

For any time dependant vector  $\mathbf{u}$  measured in a body-fixed coordinate system rotating with angular velocity  $\omega$ , the absolute rate of change is

$$\frac{d\mathbf{u}}{dt} = \left. \frac{d\mathbf{u}}{dt} \right|_{\text{rel}} + \omega \times \mathbf{u}$$

where the first term in the right-hand side denotes the rate of change of the vector seen in the body-fixed coordinate system.

# Transient beteende

Använder ett koordinatsystem som är fixt i förhållande till bilen.

Första ordningens approximation av hastighetsändringen i x-led:

$$(V_x + \Delta V_x) \cos \Delta\theta - (V_y + \Delta V_y) \sin \Delta\theta - V_x \sim \Delta V_x - V_y \Delta\theta$$

Dela med  $\Delta t$  och låt  $\Delta t$  gå mot noll:

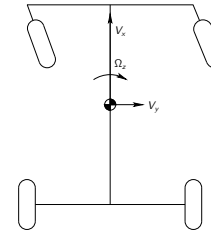
$$a_x = \frac{dV_x}{dt} - V_y \frac{d\theta}{dt} = \dot{V}_x - V_y \Omega_z$$

På samma sätt fås i y-led:

$$(V_y + \Delta V_y) \cos \Delta\theta + (V_x + \Delta V_x) \sin \Delta\theta - V_y \sim \Delta V_y + V_x \Delta\theta$$

$$a_y = \frac{dV_y}{dt} + V_x \frac{d\theta}{dt} = \dot{V}_y + V_x \Omega_z$$

# Stability of a shopping cart: Body-fixed coordinate system



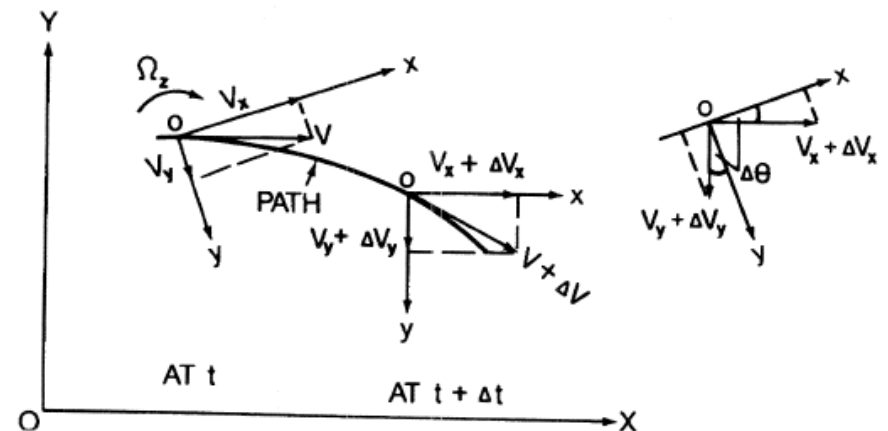
With the velocity vector as  $\mathbf{u}$  and the system rotating around the z-axis, (pointing downwards) with angular velocity  $\Omega_z$ , the relation

$$\frac{d\mathbf{u}}{dt} = \left. \frac{d\mathbf{u}}{dt} \right|_{\text{rel}} + \omega \times \mathbf{u}$$

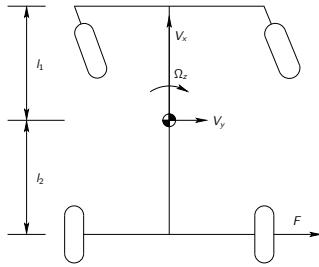
becomes

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \dot{V}_x \\ \dot{V}_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_z \end{pmatrix} \times \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \dot{V}_x - \Omega_z V_y \\ \dot{V}_y + \Omega_z V_x \\ 0 \end{pmatrix}$$

# Figur 5.18



## Stability of a shopping cart



With  $a_x = \dot{V}_x - \Omega_z V_y$  and  $a_y = \dot{V}_y + \Omega_z V_x$  the plane equations of motion for the shopping cart become

$$m(\dot{V}_x - \Omega_z V_y) = 0$$

$$m(\dot{V}_y + \Omega_z V_x) = F$$

$$I_z \dot{\Omega}_z = -l_2 F$$

It is assumed that the side velocity at the rear is equal to zero, i.e.,

$$V_y - l_2 \Omega_z = 0$$

## Stability of a shopping cart

The solution of the differential equation  $(I_z + ml_2^2)\dot{\Omega}_z + ml_2 V_x \Omega_z = 0$  is

$$\Omega_z = e^{-t/\tau}, \quad \tau = \frac{I_z + ml_2^2}{ml_2 V_x}$$

It can be seen that the yaw rate  $\Omega_z$  tends to zero as time tends to infinity, if  $V_x > 0$ . However, this does not imply that the direction of the cart will return to the direction the cart was traveling in before the perturbation.

If the basic motion is defined as  $\Omega_z = 0$  and  $V_y = 0$ , then the perturbed motion approaches the basic motion and the system is stable.

Above it was assumed that the cart was traveling forward ( $V_x > 0$ ). If the cart is pushed backwards ( $V_x < 0$ ) the cart becomes unstable ( $\tau < 0$ ).

## Stability of a shopping cart

The three equations

$$\begin{aligned} I_z \dot{\Omega}_z &= -l_2 F \\ m(\dot{V}_y + \Omega_z V_x) &= F \\ V_y &= l_2 \Omega_z \end{aligned}$$

give

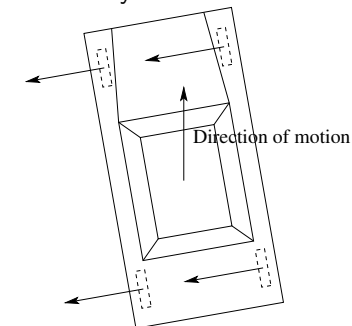
$$I_z \dot{\Omega}_z = -l_2 F = -l_2 m(\dot{V}_y + \Omega_z V_x) = -l_2 m(l_2 \dot{\Omega}_z + \Omega_z V_x)$$

which can be rewritten as

$$(I_z + ml_2^2)\dot{\Omega}_z + ml_2 V_x \Omega_z = 0$$

## Stability: Question of today

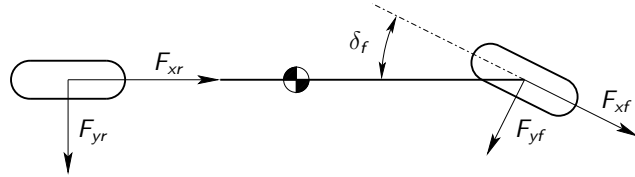
Now, we shall study the stability of the car after a disturbance.



We have seen in the previous lecture that the car will tend to turn in different directions depending on the sign of the understeer gradient  $K_{us}$ . Today's question is whether the car will tend to return to the basic motion, that we define in the same way as before to be  $\Omega_z = 0$  and  $V_y = 0$ .

## Stability: Equations of motion

The forces acting on the vehicle:



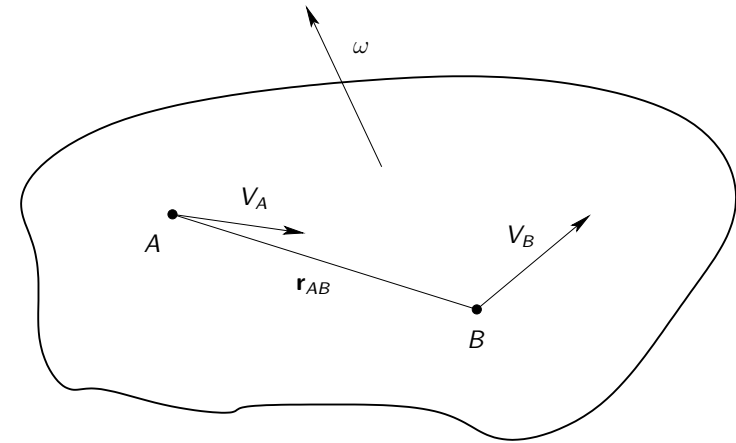
Equations of motions:

$$\begin{aligned} m(\dot{V}_x - V_y \Omega_z) &= F_{xf} \cos \delta_f + F_{xr} - F_{yf} \sin \delta_f \\ m(\dot{V}_y + V_x \Omega_z) &= F_{yr} + F_{yf} \cos \delta_f + F_{xf} \sin \delta_f \\ I_z \dot{\Omega}_z &= l_1 F_{yf} \cos \delta_f - l_2 F_{yr} + l_1 F_{xf} \sin \delta_f \end{aligned}$$

The left-hand side is the same as for the shopping cart and  $V_x$  and  $V_y$  are the velocities in the body-fixed coordinate system.

## Stability: Kinematics

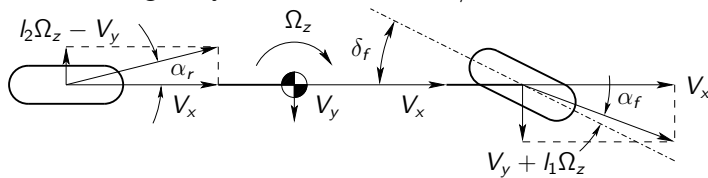
In general:



$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{AB}$$

## Stability: Kinematics

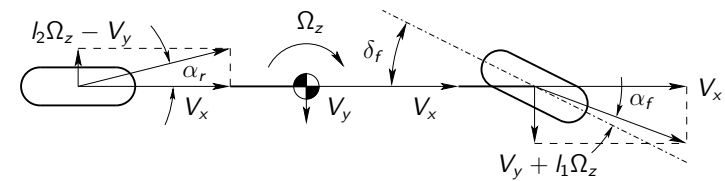
With  $A$  at center of gravity and  $B$  at the front/rear wheel:



$$\text{Front wheel: } \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_z \end{pmatrix} \times \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} V_x \\ V_y + l_1 \Omega_z \\ 0 \end{pmatrix}$$

$$\text{Rear wheel: } \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_z \end{pmatrix} \times \begin{pmatrix} -l_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} V_x \\ V_y - l_2 \Omega_z \\ 0 \end{pmatrix}$$

## Stability: Kinematics



Kinematics from the figure

$$\begin{aligned} \alpha_f &= \delta_f - \frac{l_1 \Omega_z + V_y}{V_x} \\ \alpha_r &= \frac{l_2 \Omega_z - V_y}{V_x} \end{aligned}$$

Model for tire forces

$$\begin{aligned} F_{yf} &= 2C_{\alpha f} \alpha_f \\ F_{yr} &= 2C_{\alpha r} \alpha_r \end{aligned}$$

From now it will be assumed that the longitudinal velocity  $V_x$  is constant and will play the role of a parameter in the analysis.

## Stability

From the previous slides:

$$m(\dot{V}_y + V_x \Omega_z) = F_{yr} + F_{yf} \cos \delta_f + F_{xf} \sin \delta_f$$

$$I_z \dot{\Omega}_z = l_1 F_{yf} \cos \delta_f - l_2 F_{yr} + l_1 F_{xf} \sin \delta_f$$

$$F_{yf} = 2C_{\alpha f} \left( \delta_f - \frac{l_1 \Omega_z + V_y}{V_x} \right), \quad F_{yr} = 2C_{\alpha r} \left( \frac{l_2 \Omega_z - V_y}{V_x} \right)$$

Can be rearranged in the form

$$m\dot{V}_y + \underbrace{\left[ \frac{2C_{\alpha f} + 2C_{\alpha r}}{V_x} \right]}_{a_1} V_y + \underbrace{\left[ mV_x + \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x} \right]}_{a_2} \Omega_z = 2C_{\alpha f} \delta_f(t)$$

$$I_z \dot{\Omega}_z + \underbrace{\left[ \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x} \right]}_{a_3} V_y + \underbrace{\left[ \frac{2l_1^2 C_{\alpha f} + 2l_2^2 C_{\alpha r}}{V_x} \right]}_{a_4} \Omega_z = 2l_1 C_{\alpha f} \delta_f(t)$$

## Stability

From the basic course in linear algebra we now that the solution of a system of linear equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

is a linear combination of vectors in the form

$$e^{\lambda t} \mathbf{X}$$

where  $\lambda$  is an eigenvalue of  $\mathbf{A}$  and  $\mathbf{X}$  is the corresponding eigenvector. Assume that the solution of  $M\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} = 0$  is of the form

$$\mathbf{x}(t) = e^{st} \mathbf{X}$$

where  $s$  is a constant, possibly complex a number. Substitute the expression into the system of differential equations:

$$M\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} = \left/ \dot{\mathbf{x}} = \frac{d}{dt} e^{st} \mathbf{X} = s e^{st} \mathbf{X} \right/ = e^{st} (sM + \mathbf{A}) \mathbf{X} = 0$$

## Stability

The system

$$m\dot{V}_y + \underbrace{\left[ \frac{2C_{\alpha f} + 2C_{\alpha r}}{V_x} \right]}_{a_1} V_y + \underbrace{\left[ mV_x + \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x} \right]}_{a_2} \Omega_z = 2C_{\alpha f} \delta_f(t)$$

$$I_z \dot{\Omega}_z + \underbrace{\left[ \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x} \right]}_{a_3} V_y + \underbrace{\left[ \frac{2l_1^2 C_{\alpha f} + 2l_2^2 C_{\alpha r}}{V_x} \right]}_{a_4} \Omega_z = 2l_1 C_{\alpha f} \delta_f(t)$$

can be written in matrix form

$$M\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} = \mathbf{u}(t)$$

where

$$M = \begin{bmatrix} m & 0 \\ 0 & I_z \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 2C_{\alpha f} \delta_f(t) \\ 2l_1 C_{\alpha f} \delta_f(t) \end{bmatrix}$$

## Stability

The relation

$$e^{st} (sM + \mathbf{A}) \mathbf{X} = 0$$

has to be fulfilled for all values of  $t$  and the only possibility is that the vector  $\mathbf{X}$  is the solution of the homogeneous system of equations

$$(sM + \mathbf{A}) \mathbf{X} = 0$$

A non-trivial solution ( $\mathbf{X} \neq 0$ ) exists if and only

$$\det(sM + \mathbf{A}) = 0$$



## Stability

To summarize the discussion. The procedure to calculate the solutions of the form

$$\mathbf{x}(t) = e^{st} \mathbf{X}$$

of the system of differential equations

$$M\dot{\mathbf{x}} + A\mathbf{x} = 0$$

is to first determine the eigenvalues  $s$  by solving the characteristic equation

$$\det(sM + A) = 0$$

and then for each eigenvalue  $s$  determine the corresponding eigenvectors by solving system of linear equations

$$(sM + A)\mathbf{X} = 0$$

## Stability

The system is stable if and only if the solutions of

$$\det(sM + A) = ml_z s^2 + (l_z a_1 + ma_4)s + (a_1 a_4 - a_2 a_3) = 0$$

are all in the left half of the complex plane. The solutions of a second order equation

$$s^2 + ps + q = 0$$

is given by the formula

$$s = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

It can be seen that solutions are all in the left half of the complex plane if and only if  $p > 0$  and  $q > 0$ . It is easy to verify that  $ml_z > 0$ , and  $l_z a_1 + ma_4 > 0$ . Hence, stability depends only on the sign of the expression  $a_1 a_4 - a_2 a_3$ .

## Stability

With

$$M = \begin{bmatrix} m & 0 \\ 0 & I_z \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

where  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  was defined by

$$m\dot{V}_y + \underbrace{\left[ \frac{2C_{\alpha f} + 2C_{\alpha r}}{V_x} \right]}_{a_1} V_y + \underbrace{\left[ mV_x + \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x} \right]}_{a_2} \Omega_z = 2C_{\alpha f} \delta_f(t)$$

$$I_z \dot{\Omega}_z + \underbrace{\left[ \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x} \right]}_{a_3} V_y + \underbrace{\left[ \frac{2l_1^2 C_{\alpha f} + 2l_2^2 C_{\alpha r}}{V_x} \right]}_{a_4} \Omega_z = 2l_1 C_{\alpha f} \delta_f(t)$$

the characteristic equation becomes

$$\det(sM + A) = \det \begin{bmatrix} sm + a_1 & a_2 \\ a_3 & sl_z a_4 \end{bmatrix}$$

$$= ml_z s^2 + (l_z a_1 + ma_4)s + (a_1 a_4 - a_2 a_3) = 0$$

## Stability

It can be shown that

$$a_1 a_4 - a_2 a_3 > 0$$

where  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  was defined by

$$m\dot{V}_y + \underbrace{\left[ \frac{2C_{\alpha f} + 2C_{\alpha r}}{V_x} \right]}_{a_1} V_y + \underbrace{\left[ mV_x + \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x} \right]}_{a_2} \Omega_z = 2C_{\alpha f} \delta_f(t)$$

$$I_z \dot{\Omega}_z + \underbrace{\left[ \frac{2l_1 C_{\alpha f} - 2l_2 C_{\alpha r}}{V_x} \right]}_{a_3} V_y + \underbrace{\left[ \frac{2l_1^2 C_{\alpha f} + 2l_2^2 C_{\alpha r}}{V_x} \right]}_{a_4} \Omega_z = 2l_1 C_{\alpha f} \delta_f(t)$$

is equivalent to the condition

$$L + \frac{V_x^2}{g} \left( \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} \right) = L + \frac{V_x^2}{g} K_{us} > 0$$

## Stability: Answer to the Question of the day

The condition

$$L + \frac{V_x^2}{g} \left( \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} \right) = L + \frac{V_x^2}{g} K_{us} > 0$$

can only be violated if

$$K_{us} < 0$$

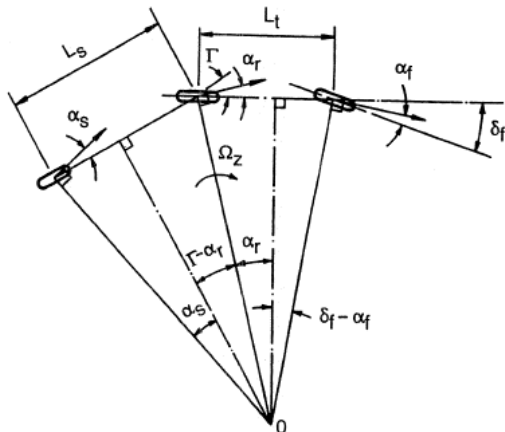
i.e. the car is understeer and

$$V \geq \sqrt{\frac{gL}{-K_{us}}} = V_{crit}$$

Answer to the question:

**The car is unstable if and only if  $K_{us} < 0$  and  $V \geq V_{crit}$ .**

## Kurvtagning: Figur 5.26



## Dragbil med semi-trailer: Kurvtagning

Figur 5.26 visar en förenklad modell.

Styrvinkel

$$\delta_f = \frac{L_t}{R} + \underbrace{\left( \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} \right)}_{K_{us,t}} \frac{V^2}{gR}$$

Vinkeln mellan dragbil och semi-trailer

$$\Gamma = \frac{L_s}{R} + \underbrace{\left( \frac{W_r}{C_{\alpha r}} - \frac{W_s}{C_{\alpha s}} \right)}_{K_{us,s}} \frac{V^2}{gR}$$

Förstärkning:

$$\frac{\Gamma}{\delta_f} = \frac{L_s/R + K_{us,s}(V^2/gR)}{L_t/R + K_{us,t}(V^2/gR)}$$

## Dragbil med semi-trailer: Fall 1

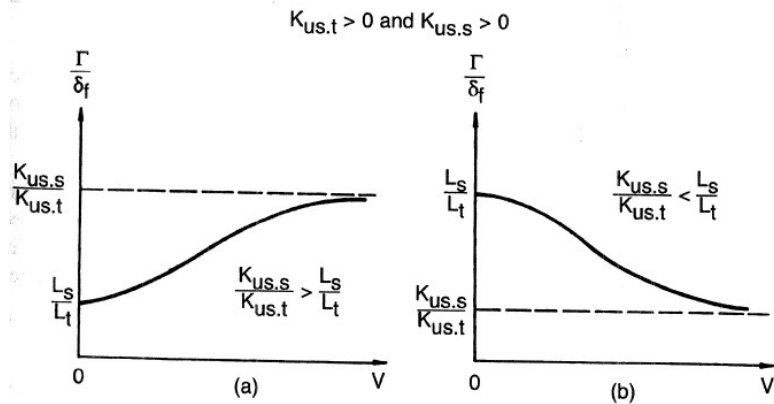
$K_{us,t}$  och  $K_{us,s}$  är båda positiva.

Förstärkningen  $\Gamma/\delta_f$  är alltid positiv.

Förstärkningen är växande om  $K_{us,s}/K_{us,t} > L_s/L_t$  och avtagande om  $K_{us,s}/K_{us,t} < L_s/L_t$ .

Se figur 5.27.

## Kurvtagning: Figur 5.27



## Dragbil med semi-trailer: Fall 2

$K_{us,t}$  är positiv och  $K_{us,s}$  är negativ.

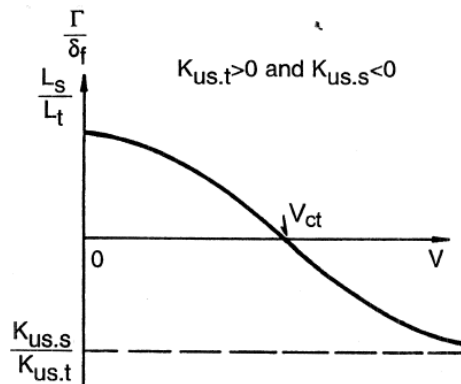
För hastigheter över

$$V_{ct} = \sqrt{\frac{gL_s}{-K_{us,s}}}$$

är förstärkningen negativ.

Se figur 5.28.

## Kurvtagning: Figur 5.28



## Dragbil med semi-trailer: Fall 3

$K_{us,t}$  är negativ och  $K_{us,s}$  är positiv.

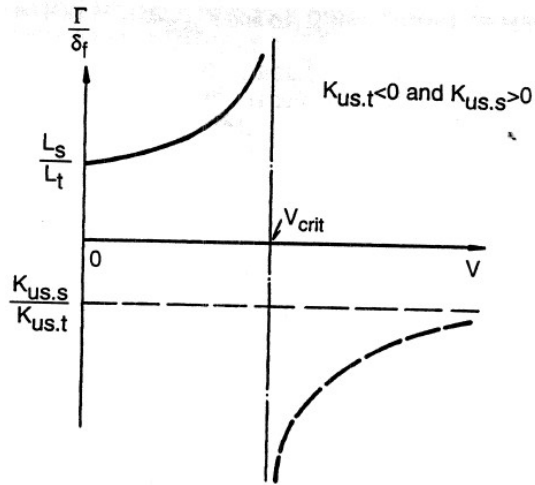
När hastigheten närmar sig det kritiska värdet

$$V_{crit} = \sqrt{\frac{gL_t}{-K_{us,t}}}$$

går förstärkningen mot oändligheten.

Bilen fälls ihop som en fällkniv ("Jackknifing"). Se figur 5.29.

## Kurvtagning: Figur 5.29



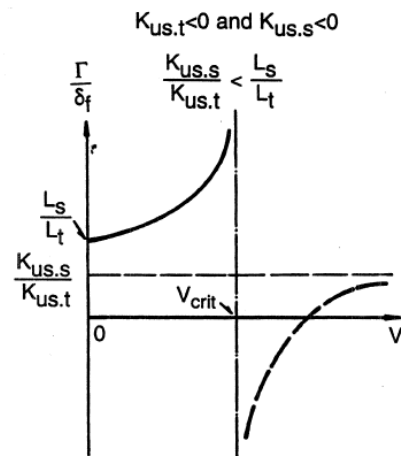
## Dragbil med semi-trailer: Fall 4

$K_{us,t}$  och  $K_{us,s}$  är båda negativa och  $K_{us,s}/K_{us,t} < L_s/L_t$ .

Det händer samma sak som i föregående fall.

Se figur 5.30.

## Kurvtagning: Figur 5.30



## Dragbil med semi-trailer: Fall 5

$K_{us,t}$  och  $K_{us,s}$  är båda negativa och  $K_{us,s}/K_{us,t} > L_s/L_t$ .

Förstärkningen blir negativ för hastigheter över den karakteristiska hastigheten

$$V_{ct} = \sqrt{\frac{gL_s}{-K_{us,s}}}$$

När hastigheten närmar sig den kritiska hastigheten

$$V_{crit} = \sqrt{\frac{gL_t}{-K_{us,t}}}$$

går kvoten mot  $-\infty$  och trailern svänger ut ("trailer swing").

Se figur 5.31.

# Kurvtagning: Figur 5.31

