

# An extremum seeking approach to parameterised loop-shaping control design

Chih Feng Lee\* Sei Zhen Khong\*\* Erik Frisk\*  
Mattias Krysander\*

\* *Division of Vehicular Systems, Department of Electrical Engineering,  
Linköping University, SE-581 33 Linköping, Sweden. (e-mail:  
chih.feng.lee@liu.se, {frisk,matkr}@isy.liu.se).*

\*\* *Department of Automatic Control, Lund University, SE-221 00  
Lund, Sweden. (e-mail: seizhen@control.lth.se).*

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**Abstract:** An approach to loop-shaping feedback control design in the frequency domain via extremum seeking is proposed. Both plants and controllers are linear time-invariant systems of possibly infinite dimension. The controller is assumed to be dependent on a finite number of parameters. Discrete-time global extremum seeking algorithms are employed to minimise the difference between the desired loop shape and the estimate of the present loop shape by fine-tuning the controller parameters within a sampled-data framework. The sampling period plays an important role in guaranteeing global practical convergence to the optimum. A case study on PID control tuning is presented to demonstrate the applicability of the proposed method.

*Keywords:* Global extremum seeking, loop-shaping, parameterised controller, frequency response estimation

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## 1. INTRODUCTION

Designing a robust controller with good reference tracking and disturbance rejection performances in the face of modelling mismatch can be a challenging task. One renowned control synthesis technique for such a task is loop-shaping which, when properly utilised, achieves robust feedback stability and performance for a given linear time-invariant (LTI) plant (Doyle et al., 1992; McFarlane and Glover, 1992; Vinnicombe, 2001). A requirement to applying the loop-shaping techniques is the availability of plant models. Nevertheless, useful plant models are often not known in many practical applications. This paper attempts to address this issue via the use of extremum seeking control (Ariyur and Krstić, 2003; Killingsworth and Krstic, 2006; Zhang and Ordóñez, 2011).

It is demonstrated that the global extremum seeking framework of Khong et al. (2013b) is suited for performing parameterised loop-shaping controller design for single-input single-output (SISO) plants. Both the plant and compensator are allowed to be infinite-dimensional (Curtain and Zwart, 1995) — for example, modelled by partial differential equations or contain time-delay terms. The compensator is presumed to be a function of a finite number of parameters and extremum seeking control is employed to drive the set of parameters to one that gives rise to a desired loop shape. By probing the loop or return ratio with appropriate signals and collecting the corresponding output measurements, the loop transfer function within the interested frequency range can be estimated. This can then be utilised for controller tuning within a periodic sampled-data framework. In particular, extremum seeking can be carried out with the cost function to be minimised being some distance measure between the achieved and

desired loop shape in the frequency domain without exploiting models of the plant and controller.

In the proposed loop-shaping procedure, the loop transfer function is estimated via the use of the Fourier transform, which in turn is approximated using the Fast Fourier Transform (FFT). This inherently introduces errors to the estimation process. It is shown that under certain robustness condition on the extremum seeking algorithm, global practical convergence to the desired loop shape is achievable by adjusting the user-designed sampling period. To illustrate the results, a case study involving the self-tuning of PID compensator parameters to attain a desired loop shape is presented.

This paper evolves along the following lines. First, the general class of systems considered in (Khong et al., 2013b) is reviewed in Section 2. Extremum seeking algorithms are discussed in Section 3 and they are established to give rise to non-local practical convergence to global extrema within a sampled-data framework from (Khong et al., 2013b). In Section 4, the loop-shaping control design problem is examined and established to be special cases of the class of systems in Section 2, depending on the type of frequency response estimation methods used. Extremum seeking methods presented in Section 3 are then shown to be applicable to the loop-shaping problem. In Section 5, the theoretical developments are furnished with a case study in self-tuning PID controller design.

## 2. DYNAMICAL SYSTEMS

It is demonstrated in Khong et al. (2013b) that the extremum seeking framework therein can accommodate a rich class of nonlinear time-invariant systems that are pos-

sibly distributed-parameter and may admit complicated attractors. These are briefly reviewed in this section. To begin with, the following notation is introduced.

A function  $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class- $\mathcal{K}$  (denoted  $\gamma \in \mathcal{K}$ ) if it is continuous, strictly increasing, and  $\gamma(0) = 0$ . If  $\gamma$  is also unbounded, then  $\gamma \in \mathcal{K}_{\infty}$ . A continuous function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class- $\mathcal{KL}$  if for each fixed  $t$ ,  $\beta(\cdot, t) \in \mathcal{K}$  and for each fixed  $s$ ,  $\beta(s, \cdot)$  is decreasing to zero (Khalil, 2002). The Euclidean norm is denoted  $\|\cdot\|$ . Given any scalar transfer function  $X$  that is continuous on  $j\mathbb{R} \cup \{\infty\}$ , let  $\|X\|_{\infty} := \sup_{\omega \in \mathbb{R}} |X(j\omega)|$ .

Let  $\mathcal{X}$  be a Banach space whose norm is denoted  $\|\cdot\|$ . Given any subset  $\mathcal{Y}$  of  $\mathcal{X}$  and a point  $x \in \mathcal{X}$ , define the distance of  $x$  from  $\mathcal{Y}$  as  $\|x\|_{\mathcal{Y}} := \inf_{a \in \mathcal{Y}} \|x - a\|$ . Also let

$$\mathcal{U}_{\epsilon}(\mathcal{Y}) := \{x \in \mathcal{X} \mid \|x\|_{\mathcal{Y}} < \epsilon\}.$$

*Definition 2.1.* Let the state of a time-invariant dynamical system be represented by  $x : \mathbb{R}_{\geq 0} \rightarrow \mathcal{X}$ , where  $\mathcal{X}$  is a Banach space with norm  $\|\cdot\|$ . The input to and output of the system are denoted, respectively, by  $u : \mathbb{R}_{\geq 0} \rightarrow \Omega \subset \mathbb{R}^m$  and  $y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ . The set  $\Omega$  denotes the input space of interest, and is taken to be a compact subset of  $\mathbb{R}^m$ . Compactness is not a stringent assumption given the ubiquity of control input saturation constraints in physical systems (Khalil, 2002). Given any  $u \in \Omega$  and  $x_0 \in \mathcal{X}$ , let  $x(\cdot, x_0, u)$  be the state of the dynamical system starting at  $x_0$  with input  $u$ .

*Assumption 2.2.* Given a system described in Definition 2.1, the following hold:

- (i) There exists a function  $\mathcal{A}$  mapping from  $\Omega$  to subsets of  $\mathcal{X}$  such that for each constant  $u \in \Omega$ ,  $\mathcal{A}(u)$  is a nonempty closed set and a *global attractor* (Ruelle, 1989):
  - (a) Given any  $x_0 \in \mathcal{X}$  and  $\epsilon > 0$ , there exists a sufficiently large  $t > 0$  such that  $x(t, x_0, u) \in \mathcal{U}_{\epsilon}(\mathcal{A}(u))$ ;
  - (b) If  $x(t_0, x_0, u) \in \mathcal{A}(u)$ , then  $x(t, x_0, u) \in \mathcal{A}(u)$  for all  $t \geq t_0$ ;
  - (c) There exists no proper subset of  $\mathcal{A}(u)$  having the first two properties above.

Furthermore,

$$\sup_{u \in \Omega} \sup_{x \in \mathcal{A}(u)} \|x\| < \infty. \quad (1)$$

- (ii) For each triplet  $(\epsilon_1, \epsilon_2, \Delta)$  of strictly positive real numbers, there exists a waiting time  $T > 0$  such that if  $\|x_0\|_{\mathcal{A}(u)} \leq \Delta$ ,

$$\|x(t, x_0, u)\|_{\mathcal{A}(u)} \leq \epsilon_1 \|x_0\|_{\mathcal{A}(u)} + \epsilon_2 \quad \forall t \geq T, u \in \Omega.$$

- (iii) There exists a locally Lipschitz function  $h : \mathcal{X} \rightarrow \mathbb{R}$  such that the system output

$$y(t) = h(x(t, x_0, u)) \quad \forall t \geq 0$$

for any constant input  $u \in \Omega$  and  $x_0 \in \mathcal{X}$ . Moreover,  $h(x_a) = h(x_b)$  for every  $x_a, x_b \in \mathcal{A}(u)$ . Since  $\mathcal{A}(u)$  is a global attractor and  $h$  is locally Lipschitz, for any  $u \in \Omega$  and  $x_0 \in \mathcal{X}$ ,

$$\begin{aligned} Q(u) &:= \lim_{t \rightarrow \infty} h(x(t, x_0, u)) \\ &= h\left(\lim_{t \rightarrow \infty} x(t, x_0, u)\right) \\ &= h(x_l), \quad \text{for some } x_l \in \mathcal{A}(u) \end{aligned}$$

is a well-defined steady-state input-output map that is Lipschitz on  $\Omega$ .

- (iv)  $Q$  takes its global minimum value in a nonempty, compact set  $\mathcal{C} \subset \Omega$ .

Examples of systems satisfying Assumption 2.2 include finite-dimensional state-space systems with equilibria or periodic attractors; see Khong et al. (2013b) for more details. As it will be shown in Section 4, the parameterised loop-shaping controller synthesis is formulated as an extremum seeking problem involving a plant satisfying Assumption 2.2.

### 3. GLOBAL EXTREMUM SEEKING

While traditional extremum seeking methods are often only able to locate local steady-state optima (Ariyur and Krstić, 2003; Teel and Popović, 2001) unless the underlying objective function is convex, Khong et al. (2013b) establishes that within its general extremum-seeking setting, a wide range of deterministic nonconvex discrete-time optimisation algorithms may be employed for global extremum seeking, including the DIRECT method (Khong et al., 2013a) and Shubert method (Nešić et al., 2013). Such algorithms are interconnected with the plant via synchronised sampler and zero-order hold. Results from Khong et al. (2013b) are summarised in this section and then applied to the parameterised loop-shaping control problem in the next.

#### 3.1 Optimisation

Consider the optimisation problem:

$$y^* := \min_{u \in \Omega} Q(u), \quad (2)$$

where  $Q : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}$  is a Lipschitz continuous function which takes its global minimum value on  $\mathcal{C} \subset \Omega$ , i.e.  $Q(u) = y^*$  for all  $u \in \mathcal{C}$ . Let  $\Sigma$  be a discrete-time extremum seeking algorithm for (2). The output sequence  $\Sigma$  generates to probe  $Q$  is denoted  $\{u_k\}_{k=0}^{\infty}$  and in the case of precise (i.e. noiseless) sampling, the collected measurements are  $y_k = Q(u_{k-1})$ ,  $k = 1, 2, \dots$ . Define also the sequence

$$\bar{y}_N := \min_{k=1, \dots, N} y_k. \quad (3)$$

Let  $\hat{\delta}$  be a non-negative real number. It follows from the above definition that the sequence  $\{\bar{y}_k\}_{k=1}^{\infty}$  converges to the closed  $\hat{\delta}$ -neighbourhood ( $\hat{\delta}$ -ball) of  $y^*$  if for all  $\epsilon > 0$ , there exists infinitely many  $N \in \mathbb{N}$  such that

$$u_N \in \{u \in \Omega \mid |Q(u) - y^*| \leq \hat{\delta} + \epsilon\}.$$

By the Lipschitz continuity of  $Q$ , corresponding to the  $\hat{\delta}$  above, there exists a  $\delta > 0$  such that the aforementioned condition holds if a subsequence of  $\{u_k\}_{k=0}^{\infty}$  converges to the  $\delta$ -neighbourhood of  $\mathcal{C}$ . That is, for all  $\epsilon > 0$  and  $K \in \mathbb{N}$ , there exists an  $N \in \mathbb{N}$  such that  $N > K$  and

$$u_N \in \mathcal{C} + (\delta + \epsilon)\bar{\mathcal{B}}, \quad (4)$$

where  $\bar{\mathcal{B}}$  denotes the closed unit ball in  $\mathbb{R}^m$ .

In the presence of bounded additive perturbations on the measurements as illustrated in Figure 1, i.e.  $y_k = Q(u_{k-1}) + w_k$  with  $|w_k| \leq \nu$  for some  $\nu > 0$ , the following assumption is important to establish convergence of the sampled-data extremum seeking scheme to be considered in Section 3.2.

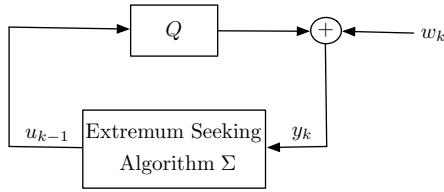


Fig. 1. Extremum seeking algorithm with noisy output measurement.

*Assumption 3.1.* Let  $\delta \geq 0$  be a small number which characterises the accuracy of convergence as in (4). The discrete-time extremum seeking algorithm  $\Sigma$  satisfies the following: Given any  $\mu > \delta$ , there exists a  $\nu > 0$  such that if  $|\hat{y}_k - Q(\hat{u}_{k-1})| \leq \nu$  for  $k = 1, 2, \dots$  and any sequence  $\{\hat{u}_k\}_{k=0}^\infty \subset \Omega$ , then for every  $\epsilon > 0$  and  $K \in \mathbb{N}$ , there exists an  $N \in \mathbb{N}$ ,  $N > K$  for which  $u_N \in \mathcal{C} + (\mu + \epsilon)\bar{\mathcal{B}}$ . In other words, the a subsequence of the output of  $\Sigma$ ,  $\{u_k\}_{k=0}^\infty$  converges to a  $\mu$ -neighbourhood of the set  $\mathcal{C}$  of global minimisers of  $Q$ .

*Remark 3.2.* Amongst others, the DIRECT algorithm (Jones et al., 1993) is an example which satisfies Assumption 3.1 (Khong et al., 2013a). Operating on a compact bound-constrained multi-dimensional domain of search

$$\Omega := \{u \in \mathbb{R}^m \mid u_i \in [a_i, b_i] \subset \mathbb{R}, i = 1, 2, \dots, m\},$$

it is intelligently balanced between local and global search. The trial points DIRECT samples in the input space always form a dense subset, whereby an output subsequence of DIRECT converges to a global extremum. This algorithm will be used for loop-shaping control design later in the paper.

### 3.2 Extremum seeking

The main sampled-data extremum seeking framework based on Khong et al. (2013b) is detailed in the following. Semi-global practical convergence to global extrema is established.

Let  $\{u_k\}_{k=0}^\infty$  be a sequence of vectors in  $\Omega$  and define the zero-order hold (ZOH) operation

$$u(t) := u_k \quad \text{for all } t \in [kT, (k+1)T) \quad (5)$$

and  $k = 0, 1, 2, \dots$ , where  $T > 0$  denotes the sampling period or waiting time. Furthermore, let the state and output of a dynamical system in Definition 2.1 with respect to the input  $u$  be respectively  $x$  and  $y$  and define the ideal periodic sampling operation  $x_k := x(kT)$ ;

$$y_k := y(kT) \quad \text{for all } k = 1, 2, \dots \quad (6)$$

Figure 2 shows an extremum seeking scheme based on a sampled-data control law with period  $T$ . The following lemma on dynamical systems is needed to establish the main result of this section.

*Lemma 3.3.* (Khong et al. (2013a)). Given any dynamical system described in Definition 2.1 that satisfies Assumption 2.2,  $\Delta > 0$ , and  $\nu > 0$ , there exists a  $T > 0$  such that for any  $\{u_k\}_{k=0}^\infty \subset \Omega$  and  $\|x_0\|_{\mathcal{A}(u_0)} \leq \Delta$ ,

$$|y_k - Q(u_{k-1})| \leq \nu \quad \text{for all } k = 1, 2, \dots,$$

where  $y_k$  is as in (6) with  $y$  being the output of the system for the input  $u$  given by (5).

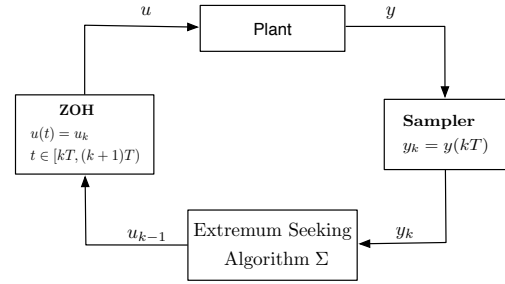


Fig. 2. Sampled-data extremum seeking control.

The main extremum seeking convergence result is stated next. The feedback configuration in Figure 2 of a dynamical plant satisfying Definition 2.1 and Assumption 2.2 and an extremum seeking algorithm  $\Sigma$  satisfying Assumption 3.1, interconnected through a  $T$ -periodic sampler (6) and a synchronised zero-order hold (5), has the following convergence property.

*Theorem 3.4.* Given any  $(\Delta, \mu)$  such that  $\Delta, \mu > \delta$ , where  $\delta \geq 0$  is given in Assumption 3.1, there exists a sampling/waiting period  $T > 0$  such that for any  $\|x_0\|_{\mathcal{A}(u_0)} \leq \Delta$ , a subsequence of  $\{u_k\}_{k=0}^\infty$  converges to  $\mathcal{C} + \mu\bar{\mathcal{B}}$ , where  $\mathcal{C}$  is the set of global minimisers for  $Q : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}$ , the steady-state input-output map of the plant, as in Assumption 2.2.

**Proof.** By Assumption 3.1, there exists a  $\nu > 0$  such that if the input to  $\Sigma$ ,  $\hat{y}$  satisfies

$$|\hat{y}_k - Q(\hat{u}_{k-1})| \leq \nu \quad (7)$$

for  $k = 1, 2, \dots$  and any  $\{\hat{u}_k\}_{k=0}^\infty \subset \Omega$ , then  $\Sigma$  generates an output  $\{u_k\}_{k=0}^\infty$  of which a subsequence converges to  $\mathcal{C} + \mu\bar{\mathcal{B}}$ . Furthermore, Lemma 3.3 ensures the existence of a sampling period  $T > 0$  such that the above-mentioned condition (7) holds for any initial plant's state  $\|x_0\|_{\mathcal{A}(u_0)} \leq \Delta$ .

## 4. LOOP-SHAPING CONTROL DESIGN

In this section, two approaches to parameterised loop-shaping control synthesis problem are shown to satisfy Definition 2.1 and Assumption 2.2. Sampled-data extremum seeking methods in Section 3 are then applied to perform the task of loop-shaping. Tuning guidelines with regards to the sampling period are provided to guarantee global practical convergence to the desired loop shape.

### 4.1 Estimation based on impulse response

Let  $C(u) : y_c \mapsto u_c$  denote a SISO LTI controller with a transfer function that is continuous on  $j\mathbb{R} \cup \{\infty\}$ , and is Lipschitz continuous in parameter  $u \in \Omega \subset \mathbb{R}^m$ .  $C(u)$  can be distributed-parameter (Curtain and Zwart, 1995), or may admit a finite-dimensional state-space realisation:

$$\begin{aligned} \dot{x}_c &= A_c(u)x_c + B_c(u)y_c, & x_c(0) &= 0 \\ u_c &= C_c(u)x_c + D_c(u)y_c, \end{aligned}$$

such that  $\text{eig}(A_c(u)) \cap j\mathbb{R} = \emptyset$  for all  $u \in \Omega$ . Let  $P : u_p \mapsto y_p$  be a SISO LTI plant that is possibly infinite-dimensional and admits a transfer function that is continuous on  $j\mathbb{R} \cup \{\infty\}$ .

Consider now the return ratio or loop gain,

$$L(u) := PC(u) : y_c \mapsto y_p.$$

In this paper, loop-shaping control design involves finding a  $u \in \Omega$  such that  $L(u)$  has a desired shape. The loop-shaping paradigm is a well-known method for synthesising a controller which gives rise to non-conservative closed-loop robustness and performance; see (McFarlane and Glover, 1992; Vinnicombe, 2001). In general, the magnitude of the loop gain is chosen to be large at frequencies where tracking performance or output disturbance rejection is important, i.e. sensitivity attenuation is required, and small at frequencies where measurement noise or modelling uncertainty could pose a problem, i.e. complementary sensitivity reduction is required. These often occur in low frequency range for the former and high frequency range for the latter. A gentle slope of the loop is also required at crossover frequency to ensure internal stability by the Bode's phase formula (Doyle et al., 1992, Chapter 7).

For any fixed  $u \in \Omega$ , the frequency response of the loop may be estimated by applying an impulse  $y_c$  to the system and taking the Fourier transform  $\mathcal{F}$  of its corresponding output/impulse response  $y_p$ , i.e.

$$L(u)(j\omega) = (\mathcal{F}y_p)(j\omega) = \lim_{T \rightarrow \infty} \int_0^T y_p(t)e^{-j\omega t} dt.$$

Let

$$x(T, u) := \int_0^T y_p(t)e^{-j\omega t} dt \quad T \geq 0.$$

The value of  $T$  serves as a waiting time: the larger  $T$  is, the closer  $x(T, u)$  will be to  $L(u)$ . Note that  $x(0) = x_0 = 0$ . It follows that there exists a  $\beta \in \mathcal{KL}$  such that for any  $u \in \Omega$ ,

$$\begin{aligned} \|x(t, x_0, u)\|_{L(u)} &= \|x(t, x_0, u) - L(u)\|_\infty \\ &\leq \beta(\|L(u)\|_\infty, t) \quad \forall t \geq 0. \end{aligned}$$

Let  $L_d(j\omega)$  be a desired loop shape and is defined in the frequency range of interests  $\omega \in [0, W]$ . Define for any function  $K$  that is continuous on  $j\mathbb{R} \cup \{\infty\}$ :

$$\|K\|_W := \sup_{\omega \in [0, W]} |K(j\omega) - L_d(j\omega)|.$$

For the purpose of extremum seeking, let the plant input be  $u$  and output be  $y(t) := \|x(t, x_0, y_c)\|_W$  for  $t \geq 0$ ; see Figure 3. As such,

$$\begin{aligned} Q(u) &:= \lim_{t \rightarrow \infty} \|x(t, x_0, y_c)\|_W \\ &= \sup_{\omega \in [0, W]} |L(u)(j\omega) - L_d(j\omega)| \end{aligned}$$

is a well-defined steady-state input-output map that is Lipschitz on  $\Omega$ . It is also clear that the global minimum is achieved when  $L(u) = L_d$ . It is straightforward to verify that the nonlinear, possibly distributed-parameter plant mapping from  $u$  to  $y$  satisfies Assumption 2.2, as required. In view of this, the following corollary to Theorem 3.4 is in order.

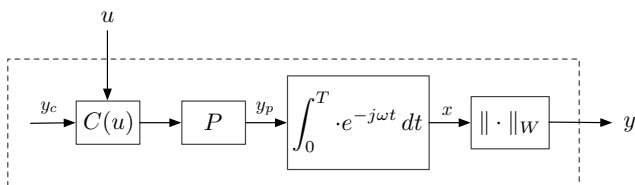


Fig. 3. Dynamical plant  $u \mapsto y$  for extremum seeking constructed from impulse response based estimation.

*Corollary 4.1.* Consider the feedback interconnection in Figure 2 of a dynamical plant  $u \mapsto y$  described by Figure 3 and an extremum seeking algorithm  $\Sigma$  satisfying Assumption 3.1. Given any  $\mu > \delta$ , where  $\delta \geq 0$  is the accuracy of  $\Sigma$  as in Assumption 3.1, there exists a sampling/waiting period  $T > 0$  such that a subsequence of  $\{u_k\}_{k=0}^\infty \subset \Omega$  converges to  $u^* + \mu\bar{\mathcal{B}}$ , where  $u^*$  satisfies  $L(u^*) = L_d$ , a prescribed loop shape. Note that the period  $T$  in both Figures 2 and 3 are the same.

#### 4.2 Estimation based on frequency sweep

Now suppose one is only interested in estimating the magnitudes of the frequency response of  $L(u)$  at frequencies  $\{\omega_1, \omega_2, \dots, \omega_N\}$ . Then the input  $y_c$  can be defined to take the form

$$y_c := a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t) + \dots + a_N \sin(\omega_N t),$$

where  $a_1, \dots, a_N > 0$ . That is,  $y_c$  sweeps constant-amplitude pure-tones/harmonics through the bandwidth of interest. By the superposition principle of linear systems, it holds that

$$y_p = b_1(u) \sin(\omega_1 t + \phi_1) + \dots + b_N(u) \sin(\omega_N t + \phi_N),$$

for some  $b_1(u), \dots, b_N(u) \in \mathbb{R}$  and  $\phi_1(u), \dots, \phi_N(u) \in \mathbb{R}$ . Thus,  $|L(j\omega_i)|$  is given by  $b_i(u)/a_i$ , for  $i = 1, \dots, N$ . Define

$$\ell(u) := [b_1(u)/a_1, \dots, b_N(u)/a_N]^T, \quad (8)$$

$x(0, u) := 0$ , and

$$x(T, u) := \left\{ \begin{array}{l} \left| \frac{\int_0^T y_p(t)e^{-j\omega_i t} dt}{\int_0^T y_c(t)e^{-j\omega_i t} dt} \right| \quad i = 1, \dots, N \end{array} \right\} \quad T > 0.$$

It follows that there exists a  $\beta \in \mathcal{KL}$  such that for any  $u \in \Omega$ ,

$$\begin{aligned} \|x(t, x_0, u)\|_{\ell(u)} &= \|x(t, x_0, u) - \ell(u)\|_2 \\ &\leq \beta(\|\ell(u)\|_2, t) \quad \forall t \geq 0. \end{aligned}$$

As before, let  $L_d(j\omega)$  be a desired loop shape and define

$$S := [|L_d(j\omega_1)|, \dots, |L_d(j\omega_N)|]^T.$$

Let the targeted plant for extremum seeking with input  $u$  have the output  $y(t) := \|x(t, x_0, u)\|_S$  for  $t \geq 0$ ; see Figure 4. Notice that

$$Q(u) := \lim_{t \rightarrow \infty} \|x(t, x_0, y_c)\|_S = \|\ell(u) - S\|_2$$

is a well-defined steady-state input-output map that is Lipschitz on  $\Omega$ . It achieves its global minimum when  $\ell(u) = S$ . As such, the nonlinear plant mapping from  $u$  to  $y$  fulfills the conditions in Assumption 2.2. Similarly to the previous subsection, one has the following corollary to Theorem 3.4.

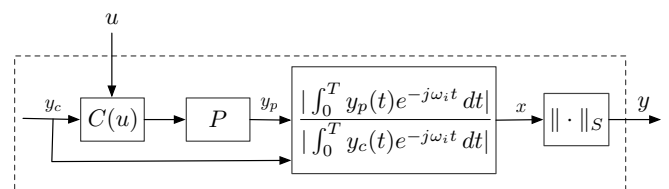


Fig. 4. Dynamical plant  $u \mapsto y$  for extremum seeking constructed from frequency sweep based estimation.

*Corollary 4.2.* Consider the feedback interconnection in Figure 2 of a dynamical plant  $u \mapsto y$  described by Figure 4 and an extremum seeking algorithm  $\Sigma$  satisfying Assumption 3.1. Given any  $\mu > \delta$ , where  $\delta \geq 0$  is

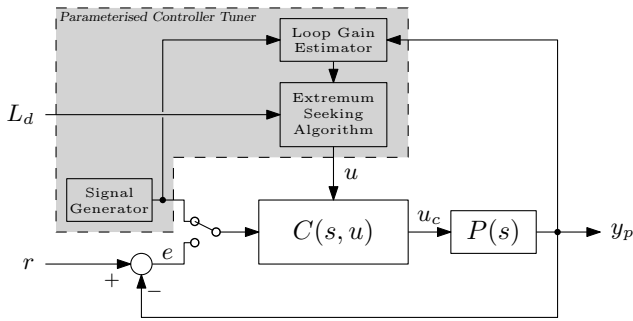


Fig. 5. Parameterised controller tuning using an extremum seeking framework.

the accuracy of  $\Sigma$  as in Assumption 3.1, there exists a sampling/waiting period  $T > 0$  such that a subsequence of  $\{u_k\}_{k=0}^{\infty} \subset \Omega$  converges to  $u^* + \mu\mathbf{B}$ , where  $u^*$  satisfies  $L(u^*) = L_d$ , a prescribed loop shape.

## 5. APPLICATION IN SELF-TUNING PID CONTROL

In this section, an example in self-tuning PID control design is shown, where the idea is to employ extremum seeking approach to tune PID parameters such that the loop gain approaches a desired loop shape. The plant is LTI and of infinite dimension. A feature of the proposed scheme is that the models of controller and plant are not needed for the tuning of controller parameters.

A plant model is borrowed from Schei (1994), where it is a stable third-order minimum phase system with time-delay, given by

$$P(s) = \frac{1 - 10s}{(1 + 60s)(1 + 20s)(1 + 20s)} e^{-10s}.$$

Note the plant is an infinite-dimensional system due to the time-delay term,  $e^{-10s}$ . A PID controller is adopted

$$C(s, u) = K_p \left( 1 + \frac{1}{T_i s} + \frac{T_d s}{1 + (T_d/N_f)s} \right) \quad (9)$$

where it is parameterised by  $u = [K_p, T_i, T_d, N_f]^T$ . Furthermore, the bounds of PID parameters are given in a compact set,  $\Omega = \{u := [K_p, T_i, T_d, N_f]^T \mid K_p = [1, 5], T_i = [1, 100], T_d = [0, 50], N_f = [0.1, 50]\}$ . The feedback structure of the plant and self-tuning controller is depicted in Figure 5.

The loop transfer function of the controlled system is

$$L(s, u) = G(s)C(s, u).$$

Suppose a desired loop shape is given by

$$L_d(s) = \frac{0.033}{s(s+2)}, \quad (10)$$

and the system behaviour within  $\omega \in [0.001, 0.01]$  Hz is of interest. Note there is no assumption made on the attainability of the desired loop shape, and exact matching to the desired loop shape may not be possible. Extremum seeking is used to find controller parameters,  $u \in \Omega$  such that  $L(s, u)$  be sufficiently close to  $L_d(s)$  within the frequency range of interest without using models of plant and controller.

A constant-amplitude pure-tones at frequencies  $\{\omega_i = i/1000 \mid i = 1, \dots, 10\}$  Hz is used to probe the open loop transfer function,  $L$ . The magnitude of  $L$  is estimated

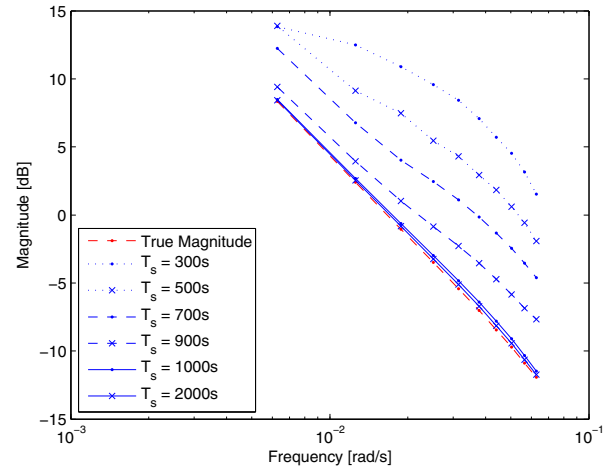


Fig. 6. True and estimated magnitudes of  $L$  with different length of  $T_s$ .

Table 1. Controller parameters obtained with different waiting periods,  $T_s$ .

Waiting Period [s]	$K_p$	$T_i$	$T_d$	$N_f$
300	1.22222	94.4444	41.6667	41.6833
700	1.22222	94.4444	41.6667	41.6833
1000	1.22222	72.2222	25	41.6833
2000	1.66667	94.4444	19.4444	41.6833

using (8), where  $a_i$  are known and  $b_i$  are calculated using fast fourier transform (FFT) algorithm (Cooley and Tukey, 1965). Figure 6 shows that the estimation accuracy of  $|L(s, u)|$  improves as the data length used for the estimation,  $T_s$  is increased.

The PID parameters are tuned using the proposed extremum seeking framework and the estimated loop shape,  $|L|$ , where the controller tunings are updated every  $T_s$ . The controller parameters obtained with different waiting periods are listed in Table 1. Note that the parameters for  $T_s = 300$  s are the same for  $T_s = 700$  s. However, Figure 7 shows that the smallest cost value obtained by extremum seeking decreases with increasing  $T_s$ . Comparing  $T_s = 300$  s and  $T_s = 700$  s, the decrease in cost is due to the improvement in  $|L|$  estimation. Figure 8 shows the loop  $L$  approaches the desired loop  $L_d$  as the waiting period is increased from 300 s to 1000 s, indicating that the longer waiting period improves the optimised result. Unit step responses obtained using the parameters in Table 1 are shown in Figure 9, where a more responsive response is obtained for  $T_s = 2000$  s.

## 6. CONCLUSIONS

An online loop-shaping design of parameterised controller is presented, where extremum seeking is adopted to tune the parameters. One motivation of the proposed method is the need to fine-tune parameters of a fixed structure controller online to accommodate for continuous changes in the plant and environment. The control design problem is formulated and shown to be a special class in the global extremum seeking framework (Khong et al., 2013b). The proposed method is applied to self-tuning PID control,

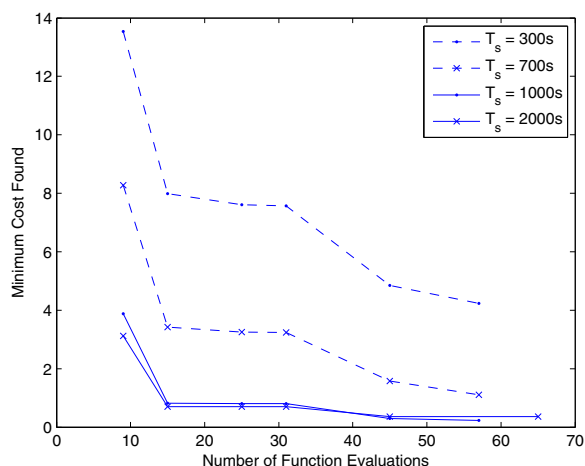


Fig. 7. Minimum cost obtained versus function count for different waiting period,  $T_s$ .

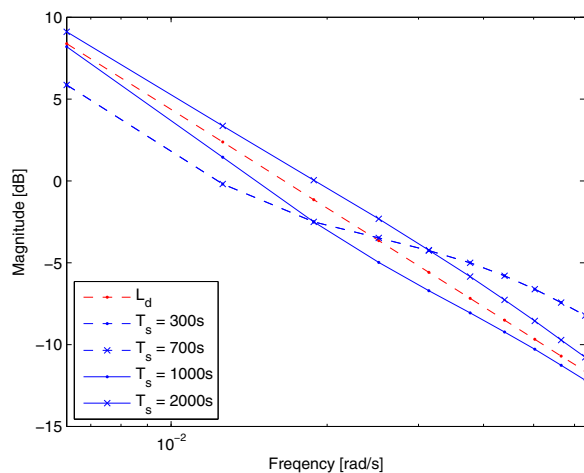


Fig. 8. Loop shape for different waiting period,  $T_s$ .

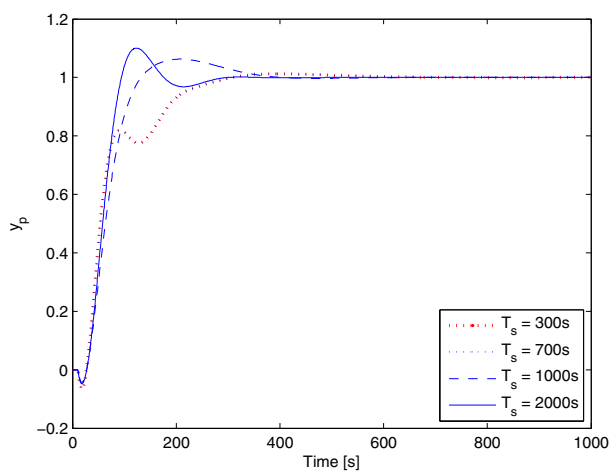


Fig. 9. Unit step responses for different waiting period,  $T_s$ .

where the magnitude plot of the compensated open-loop transfer function is close to the desired loop shape.

A novelty of the proposed method is that the models of both plant and controller are not required during the tuning process. Admittedly, if the controller model is exploited, the tuning may be accelerated. A more general design framework including controller model will be considered in the future. Additionally, alternative methods for the estimation of loop gain will be investigated.

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