Linköping Studies in Science and Technology
Thesis No. 1038

Design and Analysis of Diagnostic Systems Utilizing Structural Methods

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Linköping 2003

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ISBN 91-7373-733-X
ISSN 0280-7971
LiU-TEK-LIC-2003:37

Abstract

Today many technical processes are complex and highly integrated. When a process has failed, the complexity of the process makes it hard for humans to troubleshoot it. To facilitate troubleshooting a *diagnostic system* can supervise and alarm an operator when a fault is detected and also identify one, or several faults, that may have caused the alarm.

It is a demanding and time-consuming task to design a diagnostic system. Therefore this thesis presents algorithms and analysis methods that help and automate the design of diagnostic systems. In *model-based diagnosis* a model, in this thesis called a *diagnostic model*, of the process is used to design a diagnostic system. A diagnostic model describes the different behaviors of the *behavioral modes* of the process, which are chosen for the diagnosis task. Typical behavioral modes are the normally working mode and specified faulty working modes.

In a diagnostic system a number of *diagnostic tests* validate different models, by using observations of the process. Each test decides if the present behavioral mode of the process belongs to a subset of considered behavioral modes. If a test gives the same possible behavioral modes as the behavioral modes that together with the observations are consistent with a model, and this is true for any observation, then the test is a *strong test* for this model.

If the diagnostic model exactly describes the behaviors of the process, a goal is to design a diagnostic system such that for any observations exactly the same possible behavioral modes are given from the diagnostic system as the behavioral modes that together with the observations are consistent with the diagnostic model. A system with a set of tests so designed is called a *sound* and *complete* diagnostic system.

A key result of the thesis is, if the goal is to design a strong test for each model in a set of models, a necessary and sufficient condition for which set of models that results in a sound and complete diagnostic system. An algorithm that computes a set of models that fulfills this condition is presented. Further, an algorithm that generates a sound and complete diagnostic system for any linear static model is given.

In the two proposed algorithms for designing diagnostic systems, there is a common step that analyzes the structure of the diagnostic model, i.e. which variables that are included in each equation. The structure is used to find all minimal models of a certain type, named *minimal structurally singular* (MSS) sets of equations. A structural algorithm that finds all MSS sets in a model described by differential-algebraic equations is given. It uses a new way of handling derivatives in structural models.

Finally, the structural algorithm is applied to a large non-linear example, a part of a paper mill. In spite of the complexity of this process, a small set of tests with high isolability is successfully derived.

Acknowledgments

This work has been carried out at the department of Electrical Engineering, division of Vehicular Systems, Linköpings universitet, Sweden and in cooperation with ABB Corporate Research.

The financial support by the Swedish Agency for Innovation Systems (VINNOVA) through the center of excellence Information Systems for Industrial Control and Supervision (ISIS) is gratefully acknowledged.

I would like express my gratitude to a number of people:

My professor Lars Nielsen for letting me join this group and for supporting me during this work. The staff at Vehicular Systems for creating an excellent atmosphere.

Especially my supervisor Mattias Nyberg for encouraged me to do good research, for many interesting discussions, and for proofreading the manuscript. Erik Frisk for giving valuable feedback on the work and for solving Latex problems. Jonas Biteus for helping me with Latex as well and for supplying models for the industrial example. Jan Åslund for proofreading parts of the manuscript. Inger Klein for supporting me in ISIS. Carolina Jansson for being an excellent administrator.

Finally, I would like to thank my wife, Åsa, and my family for making the period of writing a time to remember with balance between work and leisure. Especially, I also would like to thank Gunilla, Matilda, and Åsa for giving comments on my English.

Linköping, September 2003

Mattias Krysander

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Notation

Operators and Notational Conventions

M	model, i.e. a set of relations
γ	set of models
δ	diagnostic test
Δ	diagnostic system
$\mathbb{M}$	diagnostic model
$\mathbb{M}(c)$	diagnostic model for component c
$\mathcal{R}$	rejection region or relation
$\phi(a)$	set of system behavioral modes that is implied by a
Φ	set of system behavioral modes
X_u	set of the unknown variables
Z	set of the known variables
F	set of variables describing faults
$\mathbf{x}$	vector of unknown variables
$\dot{x}$	time derivative of variable x
$\dot{e}$	equation e differentiated once with respect to time
$g^{(i)}$	i:th time derivative of g which can be an equation, a function or a variable
$\mathrm{equ}_M(X)$	equations in M that include some variable in X
$A(M, X)$	matrix A defined by the equations included in M and the variables X

$M \subseteq^* \mathbb{M}$	model M is a feasible model in $\mathbb{M}$
$\mathrm{var}_X M$	subset of variables in X that are included in some equation in M
$\mathrm{ass}\, M$	set of system behavioral modes that infer M
$\overline{\mathrm{var}}_X M$	non-differentiated variables in X that are included in some equation in M
$\widehat{\mathrm{var}} M$	variables in M with unknown derivatives
$\mathcal{O}_M^{\mathbb{M}}$	set of observations such that there exists a solution to the set of equations M in the diagnostic model $\mathbb{M}$
$\mathcal{O}_{\delta_i}^{\Delta}$	acceptance set of test δ_i in the diagnostic system Δ
$\mathrm{sol}(M, \mathbf{z})$	set of solutions to M at $\mathbf{z}$
$\mathrm{mss} M$	all MSS sets in M
M^{max}	set of most differentiated equations in M
M^{∞}	set of all equations in M together with their differentiated equations for all numbers of differentiations
M_{Φ}	The equations implied by $sys \in \Phi$
M^{+}	The (maximal) structurally overdetermined set of equations in M
M_b^*	a subset of M_b which is invalidated with exactly the same set of observations as M_b
$\beta(M, x)$	highest derivative of a non-differentiated variable x in a model M
$m(z)$	limit for variable $z \in Z$ of the order of derivative that can be considered as possible to estimate
NF	no-fault system behavioral mode
sys	true system behavioral mode
$\mathcal{G}(M, \mathrm{var}_X M)$	bipartite graph with vertices $M \cup X$ and edges $\{(e, x) \mid e \in M, x \in \mathrm{var}_X e\}$
γ_B	set of all behavioral models
γ_m	set of minimal rejectable models in $\mathbb{M}$
$\gamma_m(\mathbf{z})$	set of minimal rejectable models in $\mathbb{M}$ at $\mathbf{z}$
$\mathcal{I}^{\mathbb{M}}$,$\mathcal{I}^{\Delta}$	analytical isolability of the diagnostic model $\mathbb{M}$ and of the diagnostic system Δ respectively
$\mathcal{I}_s^{\mathbb{M}}(\gamma)$	structural isolability of the diagnostic model $\mathbb{M}$ given a set of models γ
$\mathcal{I}_s^{\Delta}$	structural isolability of the diagnostic system Δ
$\mathcal{I}_d$	desired analytical isolability
$\mathcal{I}_{sp}^{\mathbb{M}}$	primitive structural isolability of the diagnostic model $\mathbb{M}$
$\mathcal{P}^{\mathbb{M}}$, $\mathcal{P}^{\Delta}$	partial order describing the analytical isolability of the diagnostic model $\mathbb{M}$ and of the diagnostic system Δ respectively

$\mathcal{P}_s^{\mathbb{M}}(\gamma)$, $\mathcal{P}_s^{\Delta}$	partial order describing the structural isolability of the diagnostic model $\mathbb{M}$ and of the diagnostic system Δ respectively
$I^{\mathbb{M}}$, I^{Δ}	isolability matrix of the analytical isolability of the diagnostic model $\mathbb{M}$ and of the diagnostic system Δ respectively
$I_s^{\mathbb{M}}(\gamma)$, I_s^{Δ}	isolability matrix of the structural isolability of the diagnostic model $\mathbb{M}$ and of the diagnostic system Δ respectively
$\Psi(\gamma, i)$	subset of γ which have the structural isolability property i
rank(A)	rank of matrix A
srank(A)	structural rank of matrix A

Abbreviations

MSS	Minimal Structurally Singular
FDI	Fault Detection and Isolation
AI	Artificial Intelligence

1

Introduction

Today many technical processes are complex and highly integrated. When a system has failed, the complexity of the system makes it hard for humans to troubleshoot it. Since most systems nowadays have computers for control, the computers can also be used to record and calculate supporting data for repair engineers. A type of supporting system that gives possible explanations to which fault that has occurred is called a *diagnostic system*.

It is a demanding and time-consuming task to design a diagnostic system. Therefore it is valuable to automate the design of diagnostic systems. In *Model-based Diagnosis* a model of the process is used to design a diagnostic system. This thesis presents algorithms and analysis methods to design or partly design diagnostic systems given a model of the process.

This chapter starts, in Section 1.1, giving an introductionary background to model-based diagnosis. Fundamental concepts that will be used in this thesis are introduced. In Section 1.2 two research communities within model-based diagnosis is presented. Basic concepts of the two communities respectively are presented and compared to concepts used in this thesis. Then Section 1.3 summarizes the thesis and gives the main contributions. Finally, Section 1.5 contains the publications leading to this thesis.

1.1 Basic View on Model-Based Diagnosis

A diagnostic system compares expected behavior with the actual behavior. If the actual behavior deviates from the expected behavior a symptom is detected and the diagnostic system generates an alarm. By also including knowledge of faulty

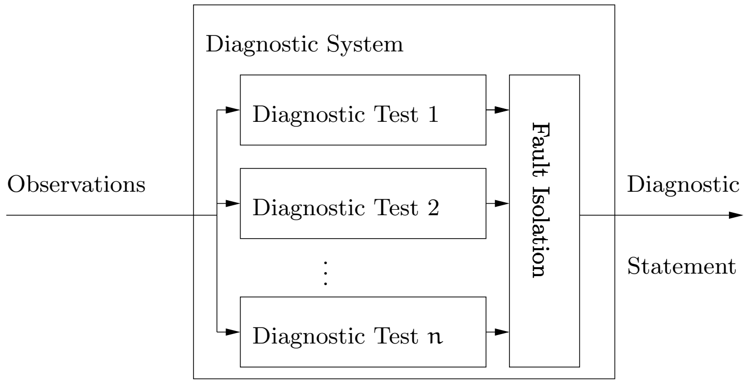

Figure 1.1 *Architecture of a diagnostic system.*

behaviors in the diagnostic system, it is able to find one or several explanations for the actual behavior. This is called *fault isolation*.

The architecture of a diagnostic system considered in this thesis is shown in Figure 1.1. A number of tests are performed using observations, also called the known variables, of the process. The goal in this thesis is diagnosis performed on-line and automatically. The observations are therefore assumed to be the actuator and sensor signals. Each test makes a binary decision. The decisions from all tests are then, in the unit "Fault Isolation" in Figure 1.1, collected into a *diagnostic statement*, i.e. a logical formula that expresses all faults that can explain the observations.

The tests are designed using a special type of process model called a *diagnostic model*. This situation is depicted in Figure 1.2. The models used for diagnosis in this thesis has an important difference compared to models used for simulation. Here a diagnostic model is actually a set of models. Apart from the model describing normal behavior, also called no-fault behavior, there are models describing different possible and pre-defined fault scenarios that are chosen to be diagnosed. Each defined behavior corresponds to a state of the process. These process states are called *behavioral modes*.

A *diagnostic test* consists of an assumption about the behavioral mode the process is working in, a scalar value computed using observations called *test quantity*, and a known set. The idea is to design the diagnostic test as follows. If the process is operating in one of the assumed behavioral modes, then the test quantity belongs to the set. If this logical relationship between the three components holds, we can conclude, if the test quantity not belongs to the set, that the process is not operating in any of the assumed behavioral modes.

The tests are derived from different parts of the diagnostic model, i.e. they are valid under different assumptions. When different sets of tests are rejected, fault isolation can be performed. The isolation capability of a diagnostic system depends

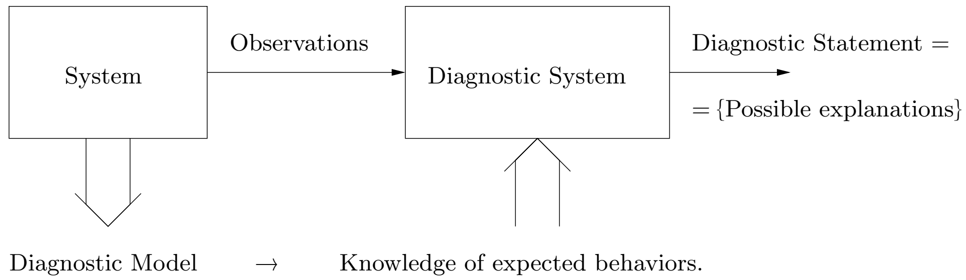

Figure 1.2 *Context of a diagnostic system. To compute a diagnostic statement a diagnostic system uses observations from the system, and knowledge of expected behavior from the diagnostic model.*

therefore on the set of tests that is used. The set of tests are on one hand defined by which parts of the diagnostic model that we want to test and on the other hand defined by the design of each test quantity and set in each diagnostic test.

Structural methods can be used to compute which models to test in order to obtain a diagnostic system with high isolation capability. Structural methods takes as input a *structural model* that describes which variables that are included in each equation.

1.2 Research within Model-based Diagnosis

In this section we connect the work presented in this thesis to existing works. Two research fields have developed model-based diagnosis (MBD) independently: the *artificial intelligence* (AI) and the *fault detection and isolation* (FDI) within control theory. The work in this thesis is influenced of ideas from both fields. An early attempt to clarify some links between the AI and FDI approach to MBD has been presented in (Cordier, Dague, Dumas, Levý, Montmain, Staroswiecki & Travé-Massuyés 2000). Section 1.2.1 and Section 1.2.2 contain the brief overview on model-based diagnosis within the AI and FDI community respectively. Basic concepts are compared to the concepts used in this thesis. Finally, Section 1.2.3 summarizes the work done on structural methods used for diagnosis.

1.2.1 The Artificial Intelligence Community

In the AI community a process contains a set of components that can fail. A diagnostic model consists of a *system description* (SD) describing the behavior of each component and their interconnections. A *conflict* is a set of components, that if they are all assumed to be working normally, resulting in an inconsistency with the SD and an observation of the behavior of the process.

In (Reiter 1987) an algorithm is developed that given conflicts calculates the set of *diagnoses*, i.e. a set of failing components that together with SD can explain the observation. The set of diagnosis is what we call the diagnostic statement. A diagnosis with more than one failing component is a *multiple fault*.

In (de Kleer & Williams 1987) the *General Diagnostic Engine* (GDE) is presented that combines model-based predictions, called local propagation, with sequential diagnosis to propose new measurement in order to isolate the faults. GDE uses an *assumption-based truth maintenance system* ATMS to keep record of assumptions and consequences. The GDE and ATMS are described in (Forbus & de Kleer 1992). The model-based predictions are used to find inconsistencies with the observed behavior and SD. The procedure of doing model-based predictions using local propagation is used instead of the testing the pre-defined set of diagnostic tests in the approach used in this thesis.

GDE handles only components that have no fault modes, i.e. they are either working normally or abnormally. In (de Kleer & Williams 1989) fault modes are introduced, i.e. each component can have several abnormal and normal modes. Considering multiple faults the number of diagnoses grows exponentially with the number of components. Adding different behavioral modes for each component also increases the number of possible diagnoses. To handle the typically very large number of diagnoses, different characterizations are presented in (de Kleer, Mackworth & Reiter 1992). In addition to good characterizations of diagnoses, different ways of estimating probabilities of the diagnoses are proposed. By estimating the probabilities of the diagnoses, it is possible to rank the diagnoses according to their probability.

A good introduction to MBD within AI is found in the collection (Hamscher, de Kleer & Console 1992) of papers. Some of the most difficult issues in MBD within the AI community are to handle noise in known variables, continuous dynamic models, and to develop models adequate for diagnosis without excessive human engineering work (de Kleer & Kurien 2003).

1.2.2 The Fault Detection and Isolation Community

In the FDI community the diagnostic models are often on a state-space form. Faults are often modeled as deviations in parameter values or unknown signals. The detection of inconsistencies is a well-studied problem within FDI. To detect that a fault has occurred, tests as described in Section 1.2 are used. A common choice of test quantity is to use a *residual*. Residuals that are suitable to use as test quantities are zero in absence of faults and non-zero for some faults. Two main approaches to construct residuals are the parity space approach (Chow & Willsky 1984, Frisk 2001, Staroswiecki & Comtet-Varga 2001) and the observer based approach (Nikoukhah 1998). Using a set of residuals, fault isolation is performed knowing which residuals that are sensitive to each fault (Gertler 1998).

Other well-studied areas within FDI are to make a sound detection in noisy environments (Basseville & Nikiforov 1993, Basseville 1998, Nyberg 1999) and to do robust design, against model errors, of the tests (Mangoubi 1998, Chen & Patton

1999).

Furthermore another studied area is data driven methods (Russell, Chiang & Braatz 2000). These methods collect large amounts of data to capture the distribution of data instead of identifying model parameters in a model suggested by physical modeling. By estimating the distribution of the data unlikely data sets are detected. If the data is translated in a direction, a sensor fault can be isolated. However process faults are difficult to isolate because the collected data only reflects the no-fault behavior.

Finally, an overview of the FDI field is presented in (Patton, Frank & Clark 2000) and some applications are presented in (Natke & Cempel 1997, Chiang, Russel & Braatz 2001).

1.2.3 Diagnosis Utilizing Structural Methods

A seminal work using structural analysis in diagnosis is (Staroswiecki & P.Declerck 1989, P.Declerck & Staroswiecki 1991) and it is briefly described next. The structure, i.e. which variables that are included in each equation, is used to find redundant equations and an elimination scheme in a systematic way in large scale models. The elimination scheme is defined using a *matching*, i.e. the equations are assigned to solve for one of their unknown variables and two equations do not solve for the same variable. Since the equations in the matching make it possible to calculate the unknown variables the remaining equations are redundant. For each redundant equation the unknown variables are substituted according to the scheme defined by the matching. Hence if some structural assumptions are fulfilled each redundant equation corresponds to a consistency relation.

A development of the work (Staroswiecki & P.Declerck 1989, P.Declerck & Staroswiecki 1991) is presented in (Blanke, Kinnaert, Lunze & Staroswiecki 2003) and (Lorentzen, Blanke & Niermann 2003, Izadi-Zamanabadi, Blanke & Katebi 2003). The matching is restricted to fulfill a *calculability* condition. The condition forbids a variable and an equation to be matched if the variable cannot always be uniquely defined with the equation assuming that all other unknown variables are known. A matching that obey this restriction is called a *causal matching*. A discussion of how the calculability restriction is done for differential equations is presented in (Lorentzen et al. 2003). It is also suggested how fault models can be included in a structural model.

Another approach that do not distinguish derivatives is given in (Ploix & Fallot 2001). The causal treatment is handled by defining input and output variables of differential equations. Only the no-fault model is considered. The aim is to find a model, i.e. a sets of equations, that can be checked for consistency. The sets found from the structural analysis are obtained using a set of structural elimination rules. An algorithm to do this elimination is not presented. In (Ploix, Touaf & Flaus 2003) an example of a complete design, including the structural step, of a diagnostic system is shown.

In (Travé-Massuyès, Escobet & Milne 2001) it is discussed how structural analysis can be used to find a minimal set of additional sensors to achieve full single fault isolation capability. The structural analysis follows the approach in (P.Declerck & Staroswiecki 1991). The contribution is the calculation of potential additional redundant relation resulting from the addition of sensors.

Within the AI field (Liegeza & Górny 2000) uses a graph obtained from simulation models in simulink/matlab. The vertices of the graph are the variables. A variable has ending directed edges from the variables that are used to calculate this variable. No fault models are used and no algorithm is described. The main goal is to show how conflict generation can be done using a dynamic model.

The AI work presented in (Pulido & Alonso 2000, Pulido & Alonso 2002) contains an important difference compared to (P.Declerck & Staroswiecki 1991). Minimal structurally overdetermined subsets of equations are found. The algorithm finds all minimal structurally overdetermined models. Then to evaluate these minimal structurally overdetermined models local propagation is used to define a substitution scheme that computes a test quantity. It is proven that this precompilation technique finds all inconsistencies that GDE finds. An advantage of pre-compilation is that differential equations can be evaluated.

Other examples of works using structural analysis that is not closely related to the work in this thesis are the work in (Pisu, Soliman & Rizzoni 2002) that uses a directed graphs to facilitate hierarchical decision making and (Bouamama, Staroswiecki, Riera & Cherifi 2000) that uses bond graphs for modeling purposes.

1.3 Summary and Contribution of the Thesis

The chapter summaries, given below, indicates the scope and the organization of the thesis. Then the main contributions are highlighted.

Chapter 2: Designing Diagnostic Systems using Diagnostic Models

In this chapter a framework for diagnostic models and diagnostic systems is presented formally. Fault models and multiple faults, fits nicely into the framework. Key concepts are formally defined such as diagnostic model, diagnostic system, and diagnostic test. It is shown how the design of a diagnostic system using a diagnostic model can be done in this framework. Two important properties that describe how the isolation capability of a diagnostic system is compared to the isolation capability of its diagnostic model are presented. These properties are called *sound* and *complete*. They are later used as a performance measure of the design of a diagnostic system given a diagnostic model. Finally the process of designing diagnostic systems is divided into two steps. In the first step a set of models is suggested to be tested such that a diagnostic system based on these tests becomes sound and complete. Then in the second step a test is designed for each suggested model to obtain a diagnostic system. In contrast to most previous works within the field on fault diagnosis, the focus of this thesis is on the first of these two steps.

Chapter 3: Structural Models and Their Properties

Structural methods can compute a set of models to test. The input for such methods is a *structural model*, i.e. a model containing only which variables that are included in each equation. Since structural models are less detailed than analytical models, the structural models can be obtained earlier in the design of a process. Since structural model can be available earlier in the development of the process, the design of a diagnostic system can start earlier. This is advantageous because then it is possible to consider the isolability aspects of for example sensor placement.

Two different structural representations of DAE systems are discussed, the *differentiated-lumped structural-model* DLSM and *differentiated-separated structural model* DSSM. The structure of a diagnostic system and a diagnostic model is defined and it is discussed how structural models can be obtained. Two advantages of analyzing structural models instead of analytical models are firstly that a structural model is easier to obtain than an analytical model, and secondly that structural analysis is computationally less complex in many cases. Fundamental structural properties are presented and finally some basic results concerning these structural properties are given.

Chapter 4: Isolability Analysis of Diagnostic Systems

In this chapter the isolation capability of a diagnostic system is analyzed. A key property of diagnostic systems is the *analytical isolability* that is formally defined. Properties of isolability are given and representations are suggested. Several reasons to make a simplified isolability analysis are presented. We propose a structural method that takes as input only a structural model. The notion of *structural isolability* is defined and an algorithm is presented that computes the structural isolability. It is proven that the structural isolability is a necessary condition for analytical isolability. Thereby it is possible to compute the structural isolability to get a limitation of the analytical isolability. We define desired isolability that is a useful and intuitive way to express one design goal of diagnostic systems. Finally, a structural method is presented. The method takes the desired isolability together with a proposed diagnostic system as inputs and computes the missing isolability properties.

Chapter 5: Analytical Characterization of Sound and Complete Diagnostic Systems

Each test in a diagnostic system is designed by the use of a model, i.e. a subset of equations of the diagnostic model. In this chapter we investigate which models that are important to design tests for, in order to get different desired properties of the diagnostic system, e.g. completeness and soundness. A key result is a necessary and sufficient condition for if a set of models can be used to design a sound and complete diagnostic system. Using this result it is possible to calculate the minimum number of tests a sound and complete diagnostic system has to have, and also to find which models to be tested.

Chapter 6: Isolability Analysis of Diagnostic Models

In this chapter we extend the two definitions analytical and structural isolability of diagnostic systems to be valid also for diagnostic models. The *analytical isolability of a diagnostic model* is the best possible analytical isolability of any diagnostic system designed using the diagnostic model. Then we present a structural method that calculates the *structural isolability of a diagnostic model*. The definition of structural isolability of a diagnostic model requires that a set of models are supposed to be tested. It is shown that if this set of models are chosen such that it corresponds to a sound and complete diagnostic system, then the structural isolability of the diagnostic model is a necessary condition for the analytical isolability of the diagnostic model. The set of models has to be chosen such that the necessary and sufficient condition presented in previous chapter is fulfilled. When the set of models is computed, the structural algorithms from Chapter 4 can be reused. It is shown how the structural algorithm easily can use additional analytical properties to compute the best possible upper limit of the analytical isolability given available knowledge. In the end of this chapter an example is presented where it is shown how the easily computed structural isolability can be used to significantly reduce the amount of computations needed to find the analytical isolability.

Chapter 7: Computing Testable Models

In Chapter 5 it is shown which sets of models that can be used to design a sound and complete diagnostic system. In this chapter we present structural methods to obtain such set of models. Models that are often especially easy to check for validity are *minimal rejectable models.* For minimal rejectable model there are observations such that no solution to exists but if any equation is removed there exist a solution to the remaining model. If a model is minimal rejectable depends on the analytical properties of the model. Since structural methods are developed the structural properties of minimal rejectable models are first presented. Using this structural characterization, algorithms are developed that compute set of models that can be used to sound and complete diagnostic systems. Finally an algorithm that is especially designed to handle large diagnostic models is presented.

Chapter 8: Structural Algorithms for Finding MSS Sets

One significant step in all the proposed structural algorithms in Chapter 7, is to find models with a structural property called *minimal structurally singular* (MSS). In this chapter we will describe algorithms that find all MSS sets for both differentiated-lumped structural-models DLSM:s and differentiated-separated structural models DSSM:s. The algorithms in this section take as input the structure of a diagnostic model. The structure of a diagnostic model can either be directly provided by the user or obtained automatically from model equations. The algorithm handles decoupling of faults. It can also take a desired isolability as input to reduce the total number of MSS sets and thereby the number of resulting tests. The desired isolability is therefore an easily understandable input that can be used to adjust the resulting diagnostic system. Furthermore an additional selecting step is described that reduce the number of tests even more without reducing the isolability.

Chapter 9: Industrial Example: A Part of a Paper Mill

In this chapter the algorithm presented in Chapter 8 is applied to a large industrial example. The example is a stock preparation and broke treatment system of a paper mill located in Australia. The model includes 4 states and most of the model equations are non-linear. In spite of the complexity of the model, the maximum isolability is calculated and a subset of MSS sets are selected that contains this isolability.

1.4 Main Contributions

Design of Diagnostic Systems:

- A necessary and sufficient condition for which sets of models that results in a sound and complete diagnostic system if a strong test is designed for each model.
- An algorithm that computes a set of models such that if a test is designed for each model a sound and complete diagnostic model is obtained.
- An algorithm that generates a sound and complete diagnostic system for any linear static diagnostic model. No assumption about linear independence is required.

Isolability Analysis:

- An algorithm that computes a structural isolability without any assumption about analytical properties.
- Theorem 4.4 and Theorem 4.5 which imply that the structural isolability can be improved if the largest set of behavioral modes supporting each equation is chosen.
- It is shown that the amount of computations needed to calculate the analytical isolability can be significantly decreased using a structural isolability.

Structural Algorithm:

- To use the minimal structurally overdetermined models also called MSS sets as a key concept in all algorithms to find a set of testable models.
- An Algorithm that finds all MSS sets in a diagnostic model consisting of DAE and arbitrary fault models.
- Structural differentiation.

- A simplification step that reduces the complexity of finding all MSS sets.
- The output of the algorithm for finding MSS sets is easily adjusted by changing the inputs, for example by changing the desired isolability.

1.5 Publications

In the research work, leading to this thesis, the author has published the following conference papers and technical reports:

- M. Krysander and M. Nyberg (2002). Structural Analysis for Fault Diagnosis of DAE Systems Utilizing Graph Theory and MSS Sets, LiTH-R-2410, Linköping University, Linköping, Sweden.
- M. Krysander and M. Nyberg (2002). Structural Analysis utilizing MSS Sets with Application to a Paper Plant, *Proc. of the Thirteenth International Workshop on Principles of Diagnosis*, Semmering, Austria.
- M. Krysander and M. Nyberg (2002). Structural Analysis for Fault Diagnosis of DAE Systems Utilizing MSS Sets, *IFAC World Congress*, Barcelona, Spain.
- M. Krysander and M. Nyberg (2002). Fault Diagnosis utilizing Structural Analysis, *CCSSE*, Norrköping, Sweden.
- M. Nyberg and M. Krysander (2003). Combining AI, FDI, and statistical hypothesis-testing in a framework for diagnosis, *Proceedings of IFAC Safeprocess'03*, Washington, USA.
- E. Frisk, D. Düştegör, M. Krysander, and V. Cocquempot (2003). Improving fault isolability properties by structural analysis of faulty behavior models: application to the DAMADICS benchmark problem, *Proceedings of IFAC Safeprocess'03*, Washington, USA.
- M. Krysander and M. Nyberg. Fault Diagnosis utilizing Structural Analysis and MSS Sets with Application to a Paper Mill, submitted to IEEE Transactions on Systems, Man and Cybernetics.

Related work:

- J. Biteus and M. Nyberg (2003). Residual generators for DAE systems utilizing minimal subsets of model equations, *Proceedings of IFAC Safeprocess'03*, Washington, USA.

2

Designing Diagnostic Systems using Diagnostic Models

In this chapter, we first introduce the essence of what diagnosis really is and presents basic diagnosis notations used frequently throughout the thesis. In Section 2.1 the diagnostic task is stated. Further the importance of diagnosis is discussed. When a fault has occurred in a system it requires a lot of system knowledge to find an explanation. The knowledge is contained in a model of the system. A general modeling framework for diagnosis purpose is presented in Section 2.2. Then in Section 2.3 we reformulate more formally the diagnostic task using the proposed modeling framework. In Section 2.4 we describe an architecture of a diagnostic system. Finally in Section 2.5 the design principles of a diagnostic system given a model is presented. Moreover some properties of diagnostic systems are presented.

2.1 The Diagnostic Task

Generally *diagnosis* is investigation or analysis of the cause or nature of a condition, situation, or problem, e.g. diagnosis of engine trouble. Diagnosis can also be a statement or conclusion from such an analysis. In this theses diagnosis analysis is performed automatically on-line using a computer. The computations are based on a model of the behavior of the system to be diagnosed and sensor and actuator values. To emphasize the importance of a model of the system to be diagnosed the notion *model-based diagnosis* is commonly used. *The diagnostic task* is given a diagnostic model that describes the expected behavior(s), state which of a set of pre-defined behavioral modes that are possible explanations for an observation.

2.1.1 Importance of Diagnosis

Efficient diagnosis is important because for example it increases safety and reliability. Diagnosis is also important for environment protection and to improve maintenance. Diagnosis is very time-consuming and advanced to perform manually. Therefore automated methods are needed.

Automated diagnosis has become more and more important over the last two decades. The reasons for the increased importance can be divided into two parts, the needs for diagnosis has increased and the availability of diagnosis has increased.

To produce competitive products for example safety, reliability, and performance have to be continuously improved. Often when technical products are improved the systems are divided into a larger number of components where each component has a more specialized function. Increasing the number of components and increasing specialization will lead to an increased system complexity. Today many technical systems have reached the limit where no man can overview the entire system. To diagnose a system is a delicate task the requires a good system knowledge. Since the engineers are not able to know all about the systems, manual diagnosis becomes ineffective or even impossible. Hence the engineers need support to draw correct conclusions. A diagnostic system supplies information to support diagnosis or even better supplies the correct diagnosis directly.

A large computation ability is a condition for diagnosis. The second factor that has made diagnosis important is that computation capacity has increased dramatically in the last decades. The computers have low production costs and are used in almost every technical system. Since there already exists processors for control purposes in many products, little extra hardware is needed to implement diagnostic systems.

2.2 Modeling Framework

2.2.1 Diagnostic System

Diagnosis as mentioned before can have many different purposes. However independent of purpose there is a common core. Diagnostic systems diagnose systems i.e. to analyze and hopefully also draw conclusions about the cause of an observed behavior. The causes are typically of the type "The system is working normally" or "The system operates with a certain fault". The states that the system can be working in, as for example no-fault or fault x, are called *behavioral modes*. Behavioral modes expressing a faulty behavior are also called *fault modes*. The set of behavioral modes are often predefined (Gertler 1998, Reiter 1987).

To decide the cause, i.e. to decide which behavioral mode that is present, diagnostic systems contain knowledge of expected behaviors of the different behavioral modes of the system to be diagnosed (Hamscher et al. 1992). The expected behavior of each behavioral mode is then compared to the behavior of the system. In this thesis the comparison of behavior is made by predefined tests included in the diagnostic system. If one such test detects a distinguishing feature of the compared behaviors, i.e. a symptom, it can be concluded that the behavioral mode corresponding to the tested behavior is not the cause of the system behavior. Hence it must be some other behavioral mode that causes the behavior. In diagnosis it is common to divide the diagnostic task into two tasks. The first task is to detect a symptom, i.e. to detect a distinguishing feature that concludes that the system does not work normally. The second task is to find which type of fault that has occurred. Very seldom it is possible to conclude the cause. Since behavioral-modes in conflict with the observed behavior are rejected the remaining behavioral modes constitute a set of possible explanations. If the diagnostic system is designed correctly the cause belongs to the set of possible explanations. It is desirable that this set is small because then the number of suspicious causes are small. Details about the behavior of each behavioral mode are essential to get few explanations (Struss & Dressler 1992, de Kleer & Williams 1989, Nyberg 1999).

2.2.2 Behavioral Modes

Behavioral modes can be defined for components or systems. To distinguish these two types of behavioral modes, they will be called *component behavioral modes* and *system behavioral-modes* respectively. A diagnosis contains a set of possible explanations of the system behavior, i.e. a set of system behavioral-modes.

System Behavioral-modes

The set of all system behavioral-modes $\mathcal{B}$ defines the set, that all diagnoses are chosen among. Hence if an important system behavioral-mode is not defined then the diagnosis will in that case never include the correct cause. It is not unusual that a fault in some component infer another fault in another component. This type of faults when several faults are present at the same time are called *multiple faults*. A fault that only change one component behavior is called a *single fault*. All possible multiple faults have to be included in the set of all system behavioral-modes. This approach is a common approach in the FDI community (Gertler & Singer 1990, Nyberg 1999). Let us exemplify system behavioral-modes with an illustrative example that will be used throughout the thesis.

Example 2.1 A pump is pumping water into the top of a tank. The system is shown in Figure 2.1. The water flows out of the tank through a pipe connected to the bottom of the tank. The known variables are the pump input u, the measured water-level in the tank y_h, and the measured water-flow from the tank y_f.

The system behavioral-modes that are relevant for this system has turned out to be: **NF** everything is working correctly; $\mathbf{PS}_p$ pump is stuck; $\mathbf{UF}_p$ unknown pump fault; $\mathbf{C}_t$ clogging in the bottom of the tank where the pipe is connected; $\mathbf{L}_i, i \in \{p_1, p_2\}$ leakage before and after water-flow sensor; $\mathbf{UF}_i, i \in \{s_1, s_2\}$ unknown sensor fault for sensor 1 and sensor 2 respectively. These are the no-fault plus the 7 single fault system behavioral-modes.

As long as it is assumed that single faults are the only faults that can happen,

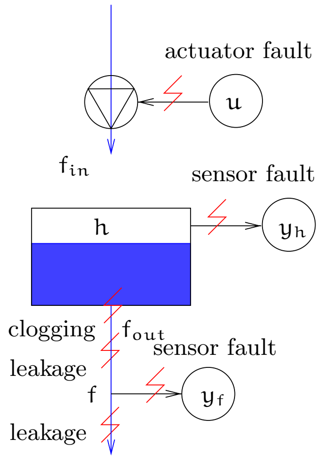

Figure 2.1 *The system to be diagnosed.*

i.e. single fault assumption, this method considering system behavioral-modes is acceptable. However, if all possible combinations of single faults are needed to be considered, this approach will be inefficient as the number of system behavioral-modes increase exponentially with the number of components.

Components and Component Behavioral-modes

In the AI community multiple faults have been extensively examined (de Kleer & Williams 1987, de Kleer et al. 1992). A common way to handle large number of system behavioral-modes is to introduce the concept of components. Component behavioral-modes are considered to be single faults, i.e. the behaviors of all fault modes have to be modeled individually. A possible choice of components is to define them according to the physical parts of the system that can be replaced. However with this principal there is different levels of granularity. The level of granularity is preferably chosen according to which faults that will be analyzed. It is preferable to divide a component into several smaller components if the component has a large number of fault modes which are multiple faults considering the smaller components. On the other hand components without any fault modes can be merged. The choice of components is clarified with the watertank example.

Example 2.2 Assume that all multiple faults are important behavioral modes for this system. To consider the entire system as one component results in a large number of fault modes as a consequence of multiple faults on a more detailed level of granularity. It is suitable to divide the system into components. To guide this division all physical components that have one or several single faults are identified. In Example 2.1 the possible faulty components are the pump, the tank, 2 pipes, and the 2 sensors. Looking at Figure 2.1 it is clear that, e.g. the pipe connecting the

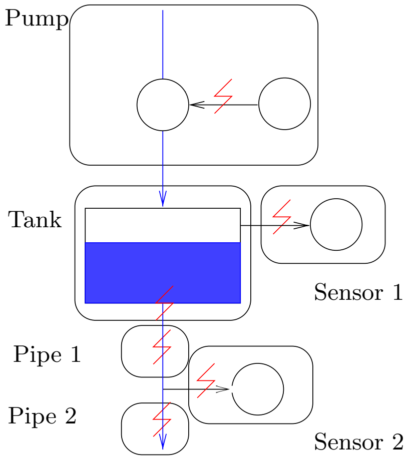

Figure 2.2 *The system divided into components.*

Table 2.1 *Component behavioral-modes for the water tank.*

component	behavioral modes
Pump	$p \in \{NF, PS, UF\}$
Tank	$t \in \{NF, C\}$
Pipe 1	$p_1 \in \{NF, L\}$
Sensor 1	$s_1 \in \{NF, UF\}$
Pipe 2	$p_2 \in \{NF, L\}$
Sensor 2	$s_2 \in \{NF, UF\}$

pump and the tank could also be considered to be another component. However, this pipe has no fault mode and is not needed for diagnostic purpose. To do the modeling as simple as possible, the lowest number of components are chosen. Therefore the pipe connecting the pump and tank is not defined as a component. The components will be called pump, tank, sensor 1, sensor 2, pipe 1, and pipe 2. The names and the corresponding physical components are seen in Figure 2.2. With this partition the component behavioral-modes are summarized in Table 2.1.

Connection between System Behavioral-modes and Component Behavioral-modes

A system behavioral-mode is a *mode assignment* such that each component has been assigned a component behavioral-mode. The system behavioral-mode will be denoted sys. Then for example the system behavioral-mode assignment

$sys = \mathbf{NF}$ in Example 2.2 means that $p = \mathsf{NF} \wedge t = \mathsf{NF} \wedge p_1 = \mathsf{NF} \wedge s_1 = \mathsf{NF} \wedge p_2 = \mathsf{NF} \wedge s_2 = \mathsf{NF}$. This will also be written as the tuple $\langle p, t, p_1, s_1, p_2, s_2 \rangle = \langle \mathsf{NF}, \mathsf{NF}, \mathsf{NF}, \mathsf{NF}, \mathsf{NF} \rangle$ or **NF**. Note that the total number of system behavioral-modes has now increased from 8 considering only single faults as in Example 2.1 to 96 adding multiple faults.

2.2.3 Diagnostic Model

In the beginning of this chapter it was stated that diagnostic systems include information of expected behavior for the different behavioral modes. The behaviors are in this thesis assumed to be expressed as a set of nonlinear differential equations and is called a *behavioral model*.

Further in the last sections the systems are divided into a set of components. The behavior of a system behavioral-mode can be obtained using a composite model of the component behavioral models engaged by this particular system behavioral-mode. In this way it is sufficient to build behavioral models for the components individually. A *component model* is the component behavioral-models for all its behavioral modes.

Component Model

A component has *internal* and *external* variables. External variables are observed variables and variables shared between connected components. Internal variables only connects different constraints inside a component. The system in Example 2.2 will in the next example be explained in the component oriented framework to illustrate the different type of variables.

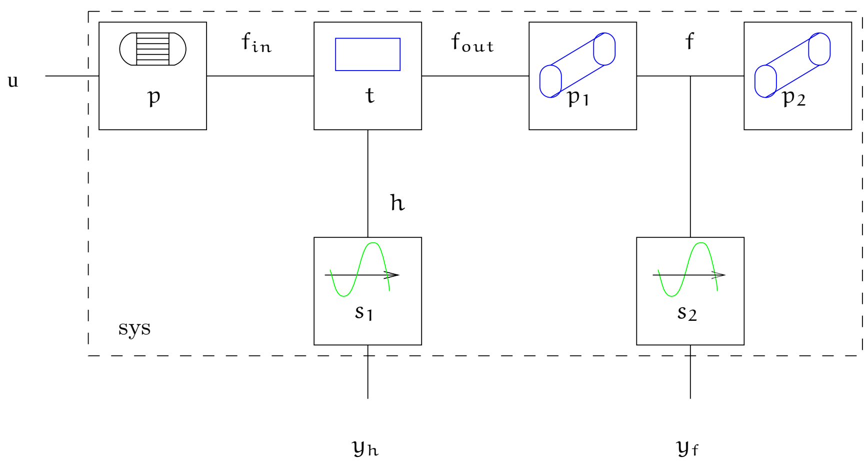

Figure 2.3 *A choice of components of the water-tank example.*

Example 2.3 A component oriented view of a model of the water tank system is shown in Figure 2.3. In this figure external variables corresponds to either connections between components or lose ends. The dashed line indicate that the system itself can be considered as a super-component. It is only the observations that are external variables with respect to the system. It is only these observations (or external variables with respect to the system) that can be used to diagnose the system.

Each component has a set of constraints that describe how the external variables are expected to be related to each other, in different behavioral modes. These constraints are assumed to be nonlinear differential equations. Assume that a component c has the following equations $\{e_{i_c}, e_{i_c+1}, \ldots, e_{i_c+n}\}$. Since these equations are going to express sometimes completely different behaviors depending on the behavioral mode, the validity of each equation depends on the assumed behavioral mode. The *behavioral mode assumption* needed to imply an equation e is described with a set of system behavioral modes denoted $\text{ass}\, e \subseteq \mathcal{B}$. Formally this is written $sys \in \text{ass}\, e \to e$. To give an example, consider the pump in the water tank example.

Example 2.4 The pump has, as previously defined in Table 2.1, three behavioral modes. For each of these behavioral modes, a detailed model has to be designed constraining the external variables of the pump, i.e. u and f_{in} as seen in Figure 2.3. When the pump is fault free, $p = \mathsf{NF}$, it holds that $f_{in} = u$, i.e. $p = \mathsf{NF} \to f_{in} = u$. The next behavioral mode is pump stuck, $p = \mathsf{PS}$. It means that independently of the applied actuator signal u the water-flow f_{in} remains 0. This is modeled as $p = \mathsf{PS} \to f_{in} = 0$. The last behavioral mode, the unknown behavior $p = \mathsf{UF}$, does not constrain the external variables and no equation can be supported in this case. The pump model is

$$\begin{aligned} p = \mathsf{NF} &\to f_{in} = u \\ p = \mathsf{PS} &\to f_{in} = 0 \end{aligned} \tag{2.1}$$

When a diagnostic model consists of several components, logical expressions is a good way to describe a set of system behavioral modes. We will use the pump in the water tank example to introduce a useful notation.

Assume that the pump is working, i.e. $p = \mathsf{NF}$, and we are interested to express the set of all system behavioral modes that says that $p = \mathsf{NF}$. It is easily realized that

$$\{\langle p, t, p_1, s_1, p_2, s_2 \rangle | \quad \begin{aligned} & p = \mathsf{NF}, t \in \{\mathsf{NF}, \mathsf{C}\}, p_1 \in \{\mathsf{NF}, \mathsf{L}\}, \\ & s_1 \in \{\mathsf{NF}, \mathsf{UF}\}, p_1 \in \{\mathsf{NF}, \mathsf{L}\}, s_2 \in \{\mathsf{NF}, \mathsf{UF}\}\} \end{aligned} \tag{2.2}$$

is the set of system behavioral modes that says that $p = \mathsf{NF}$. This set will be denoted $\phi(p = \mathsf{NF})$. Note that the result of the function ϕ is also dependent of the complete set of system behavioral modes $\mathcal{B}$ which is implicitly defined by the diagnostic model considered. This notation will also be used for logical formulas,

Table 2.2 *A diagnostic model for the water tank system.*

component	assumption	equation	
Pump	$\phi(p = NF)$	e_1 :	$u = f_{in}$
	$\phi(p = PS)$	e_2 :	$f_{in} = 0$
Tank	$\mathcal{B}$	e_3 :	$\dot{h} = f_{in} - f_{out}$
	$\phi(t = NF)$	e_4 :	$h = f_{out}^2$
	$\phi(t = C)$	e_5 :	$h = A\, f_{out}^2$
		e_6 :	$\dot{A} = 0$
Pipe 1	$\phi(p_1 = NF)$	e_7 :	$f = f_{out}$
	$\phi(p_1 = L)$	e_8 :	$f = \theta_1\, f_{out}$
		e_9 :	$\dot{\theta}_1 = 0$
Sensor 1	$\phi(s_1 = NF)$	e_{10} :	$y_h = h$
Pipe 2	$\phi(p_2 = NF)$	e_{11} :	$f = f_{int}$
	$\phi(p_2 = L)$	e_{12} :	$f = \theta_2\, f_{int}$
		e_{13} :	$\dot{\theta}_2 = 0$
Sensor 2	$\phi(s_2 = NF)$	e_{14} :	$y_f = f$

e.g. $\phi(p = NF \wedge p_1 = NF)$. When not all components are assigned a component behavioral mode as in the previous example, it is called a *partial assignment*. If a is a formula expressing a partial assignment then ϕ is defined as the set of system behavioral modes that includes the partial assignment a. The operators ass and ϕ can be used as $\text{ass}\, e_2 = \phi(p = PS)$.

Example of a Diagnostic Model

Building a model for each component in the water tank example and collecting them, the model in Table 2.2 is obtained. The diagnostic model in Table 2.2 describes many different behaviors. It will be important to have a notation for the equations valid given a system behavioral mode assumption.

Definition 2.1. *Given a diagnostic model* $\mathbb{M}$ *and a set of system behavioral-modes* Φ*, the model* M_Φ *is defined as*

$$M_\Phi := \{e | \Phi \subseteq \text{ass}\, e\} \tag{2.3}$$

If $\Phi = \{b\}$ we will also write M_b as a shorthand notation for $M_{\{b\}}$. As an example consider again the no-fault model of the model in Table 2.2, i.e. $M_{\mathbf{NF}} = \{e_1, e_3, e_4, e_7, e_{10}, e_{11}, e_{14}\}$. Next we will explain how diagnostic systems use a diagnostic model to find possible explanations for observations.

2.3 Diagnosis Utilizing Diagnostic Models

A diagnostic system uses observations from the system to be diagnosed and is designed using a diagnostic model to find possible causes. A reformulation of finding possible causes is to decide which of the pre-defined system behavioral-modes that can explain the observations. Let $b \in \mathcal{B}$ be any system behavioral mode, $\mathbf{x}$ and $\mathbf{z}$ vectors of unknown and known variables respectively. If a $\mathbb{M}$ is a static model then the diagnostic task can be formulated as:

Given an observation $\mathbf{z}$ *find all system behavioral modes* b *such that* $\exists \mathbf{x} M_b(\mathbf{x}, \mathbf{z})$.

This procedure can be repeated every time instance. If the model $\mathbb{M}$ is a set of non-linear differential-equations, then at a given time t_1 and for given observations $\{\mathbf{z}(t) | t_0 \leq t \leq t_1\}$ the diagnosis task is to find all system behavioral-modes b such that

$$\forall t \in [t_0, t_1] \exists \mathbf{x}(t) \ M_b(\mathbf{x}(t), \mathbf{z}(t)) \tag{2.4}$$

For a given b the problem of deciding if there is a trajectory $\{\mathbf{x}(t) | t_0 \leq t \leq t_1\}$ that satisfy the model M_b and the observations is a hard problem. No efficient method is available to check validity on-line when M_b consists of DAE:s, which is the case in this theses. Therefore an off-line pretreatment of the model is needed to simplify the problem to be solved on-line. A common solution especially explored in FDI community, is to derive a set of tests using the diagnostic model (Gertler & Singer 1990). The knowledge of the diagnostic model is ideally also contained in the derived tests. Then the set of derived tests can replace the diagnostic model in the diagnostic system. The advantage of deriving tests are that these tests are much more easily evaluated then directly validate the diagnostic model. Evaluation of a set of tests is possible to do on-line.

2.4 Diagnostic Systems

2.4.1 Diagnostic Tests

In statistics, theories has been developed to make correct decisions in noisy environments using tests. One method is called *statistical hypothesis tests* (Casella & R.L.Berger 1990, Berger 1985, Nyberg 2001). Choosing the recommended approach using tests, the design of a diagnostic system can be devided into two steps. The first step concerns finding a suitable set of tests. In the second step, it is convenient to use the well established theories of statistical hypothesis tests.

Each test has a scalar test quantity $T(z)$, that is a function of known variables z. Typically there are system behavioral-modes that implies that $T(z) = 0$ ideally. When $T(z) \neq 0$ the system behavioral-modes implying that $T(z) = 0$ are rejected as possible explanations. In real cases noise and model uncertainties corrupt measurements and therefore is $T(z) \approx 0$. To secure that possible explanations are not rejected, conclusions are drawn only when $T(z) \in \mathcal{R}$ where $\mathcal{R}$ is a *rejection region* such that $0 \notin \mathcal{R}$.

When $T(z) \in \mathcal{R}$ then the behavioral modes that support the test quantity are *rejected* and the number of possible explanations are reduced. This method of utilizing such a statistical framework in diagnosis has been called *structured hypothesis tests* (Nyberg 2002). In this theses noise will not be considered so a simplified test can be used. However, when noise is important the simplified tests can be replaced with standard statistical hypothesis tests.

Definition 2.2 (Diagnostic Test). *Let $\Phi_i \subseteq \mathcal{B}$ and let sys denote the true system behavioral mode. A* ***diagnostic test*** *δ_i for the null hypothesis $H_i^0 : \mathsf{sys} \in \Phi_i$ is a hypothesis test consisting of a test quantity $T_i(\mathbf{z})$ where $\mathbf{z}$ is a vector with known variables and a rejection region $\mathcal{R}_i$ such that*

$$\mathsf{sys} \in \Phi_i \rightarrow T_i(\mathbf{z}) \in \mathcal{R}_i^C \tag{2.5}$$

where $\mathcal{R}_i^C$ is the complement of $\mathcal{R}_i$

Note the index i of all quantities of the diagnostic test. This index shows for example that Φ_i is the null hypotheses behavioral-mode assumption of test i. To summarize the different parts of a diagnostic test, it consists of an assumption Φ_i, a test quantity $T_i(z)$, and a rejection region $\mathcal{R}_i$. They all have to be carefully selected to fulfill their purpose in the definition of diagnostic tests. The water tank example will be used to exemplify how these three components in a diagnostic test can be chosen.

Example 2.5 Assume that $\Phi_1 = \{\mathbf{NF}\}$ then $H_1^0 : \mathsf{sys} \in \Phi_1$. This means that $M_{\mathbf{NF}}$ are the valid equations. A subset of those are

$$\begin{array}{ll} e_4: & h = f_{out}^2 \\ e_7: & f = f_{out} \\ e_{10}: & y_h = h \\ e_{14}: & y_f = f \end{array} \tag{2.6}$$

Eliminating the unknown variables f, f_{out}, and h in equations (2.6) implies

$$y_h - y_f^2 = 0 \tag{2.7}$$

containing only known variables. From equation (2.7) a test quantity could be chosen as

$$T_1 = y_h - y_f^2 \tag{2.8}$$

Now it only remains to decide a rejection region $\mathcal{R}_1$. From the calculations it is clear that $\mathbf{NF} \rightarrow T_1 = 0$ and hence any $\mathcal{R}$ such that $0 \in \mathcal{R}_1^C$ is a valid choice. The rejection region can for example be chosen as $\mathcal{R}_1 = \{T_1 : 1 < |T_1|\}$. This choice of $\mathcal{R}_1$ is conservative. It can be seen as a precaution to not reject a valid model and hence draw wrong conclusions. The prize paid is a less sensitive test that needs more excitation to alarm.

Now diagnostic test has been defined and the next step is to understand how they are used in diagnosis. When a test is rejected, i.e. $T_i(z) \in \mathcal{R}_i$ then according to (2.5) it holds that

$$T_i \in \mathcal{R}_i \rightarrow \mathsf{sys} \in \Phi_i^C \tag{2.9}$$

The expression $sys \in \Phi_i$ becomes a *conflict* (de Kleer & Williams 1989), i.e. an expression of behavioral modes that is in conflict with the observations. For example assume that $y_h = 0$ and $y_f = 2$. Then the test quantity (2.8) is $T_1 = -4 \in \mathcal{R}_1$. This means that $sys \neq \mathbf{NF}$, i.e. a fault is detected.

2.5 Designing Diagnostic Systems

2.5.1 Designing Diagnostic Tests

As seen in Example 2.5 the design of a diagnostic test exploits a part of the diagnostic model. Let $M_i \subseteq M$ be the model used to derive test i. Now the design of a test includes choosing Φ_i, T_i, $\mathcal{R}_i$ and M_i such that expression (2.9) is fulfilled. To generalize the method in Example 2.5 Φ_i is first decided. The diagnostic model is as explained earlier on the form

$$sys \in \Phi_i \to z \in \{z|\exists x M_{\Phi_i}(x, z)\} \tag{2.10}$$

This means that $M_i \subseteq M_{\Phi_i}$ fulfills

$$sys \in \Phi_i \to z \in \{z|\exists x M_i(x, z)\} \tag{2.11}$$

The next step is to derive a test quantity T_i and a rejection region $\mathcal{R}_i$ from M_i. To be sure that expression (2.9) is fulfilled the following must hold.

$$z \in \{z|\exists x M_i(x, z)\} \to T_i(z) \in \mathcal{R}_i^C \tag{2.12}$$

If expressions (2.11) and (2.12) hold then Φ_i, T_i, and $\mathcal{R}_i$ defines a diagnostic test according to expression (2.5).

2.5.2 Diagnostic Systems

As explained earlier it is difficult or impossible to directly evaluate a diagnostic model. Therefore the model is replaced with a set of tests in the diagnostic system. A diagnostic system is in this thesis defined as follows.

Definition 2.3 (Diagnostic System). *A **diagnostic system** is a set of diagnostic tests, i.e.* $\{\delta_1, \delta_2, \ldots\}$ *together with the procedure to form a set of **candidates*** $\mathcal{C}(\mathbf{z})$ *defined as*

$$\mathcal{C}(\mathbf{z}) = \mathcal{B} \cap \bigcap_{i:H_i^0 \textit{ rejected}} \Phi_i^C \tag{2.13}$$

Candidates are all system behavioral-modes that the diagnostic system suggests as possible explains for the observation.

2.5.3 Diagnostic Model vs. Diagnostic System

The goal is to design a diagnostic system that exploit as much of the information contained in the diagnostic model as possible. For diagnosis purpose a interesting comparison concerns:

> *Given any observation, are the same system behavioral-modes consistent with the diagnostic model as with the diagnostic system?*

The candidates are the system behavioral-modes that are consistent with the diagnostic system. The system behavioral-modes that are consistent with the diagnostic model will be defined next.

Definition 2.4 (Diagnosis). *Given an observation* $\mathbf{z}$ *and a diagnostic model* $\mathbb{M}$*, a* ***diagnosis*** b *is a system behavioral-mode such that* $\exists \mathbf{x} M_b(\mathbf{x}, \mathbf{z})$.

A diagnosis is a system behavioral-mode that is consistent with the observations using the diagnostic model. The set of diagnosis given an observation $\mathbf{z}$ will be denoted $\mathcal{D}(\mathbf{z})$. For an optimal diagnostic system it holds that

$$\mathcal{C}(\mathbf{z}) = \mathcal{D}(\mathbf{z}) \tag{2.14}$$

This is an important property that can be used to compare a diagnostic system with a diagnostic model. Next we define two different properties that together imply (2.14).

Definition 2.5 (Complete). *Given a diagnostic model* $\mathbb{M}$*, a diagnostic system* Δ *is* ***complete*** *with respect to* $\mathbb{M}$ *if*

$$\forall \mathbf{z} : \mathcal{D}(\mathbf{z}) \subseteq \mathcal{C}(\mathbf{z}) \tag{2.15}$$

If all tests are designed such that expression (2.5) is fulfilled then the diagnostic system is complete. From now on we assume that all tests are designed to fulfill (2.5).

Definition 2.6 (Sound). *Given a diagnostic model* $\mathbb{M}$*, a diagnostic system* Δ *is* ***sound*** *with respect to* $\mathbb{M}$ *if*

$$\forall \mathbf{z} : \mathcal{C}(\mathbf{z}) \subseteq \mathcal{D}(\mathbf{z}) \tag{2.16}$$

The task is to design a set of diagnostic tests such that all or as much isolation capability from the diagnostic model is preserved.

2.5.4 Designing a Sound and Complete Diagnostic System

A diagnostic system should be designed for a diagnostic model $\mathbb{M}$. Let the set of all system behavioral-modes be denoted $\mathcal{B}$. A diagnostic system is sound if the following expression holds

$$\forall \mathbf{z} \forall b \in \mathcal{B} \Big(\neg \exists x M_b(x, z) \rightarrow \exists \delta_i \big((T_i(\mathbf{z}) \in \mathcal{R}_i) \wedge (b \in \Phi_i) \big) \Big) \tag{2.17}$$

Table 2.3 *A diagnostic model.*

component	assumption	equation
Sensor 1	$\phi(s_1 = \mathrm{NF})$	$e_1: \quad z_1 = x_1$
Comp	$\mathcal{B}$	$e_2: \quad x_1 = x_2^2$
Sensor 2	$\phi(s_2 = \mathrm{NF})$	$e_3: \quad z_2 = x_2$

The interpretation of (2.17) is that if the model for any system behavioral-mode is rejected, i.e. this behavioral mode is not a diagnosis, then there is a diagnostic test that has rejected its null hypothesis. Furthermore, the negated assumption of the null hypothesis implies that this particular system behavioral-mode is not a candidate. Since both the observation and the system behavioral-mode were arbitrarily chosen, any system behavioral-mode that is not a diagnosis is not a candidate either, i.e. all candidates are diagnosis. Next an example illustrates how expression (2.17) can be used to design a sound diagnostic system.

Example 2.6 Consider the small diagnostic model in Table 2.3. Using expression (2.17) all system behavioral-modes have to be analyzed. The system behavioral-modes are

$$\mathcal{B} = \{\langle s_1, s_2\rangle | \langle \mathrm{NF}, \mathrm{NF}\rangle, \langle \mathrm{NF}, \mathrm{UF}\rangle, \langle \mathrm{UF}, \mathrm{NF}\rangle, \langle \mathrm{UF}, \mathrm{UF}\rangle\} \tag{2.18}$$

The model for the first system behavioral-mode in (2.18) is $M_{\langle \mathrm{NF},\mathrm{NF}\rangle} = \{e_1, e_2, e_3\}$. Eliminating the unknown variables x_1 and x_2 gives

$$\{\mathbf{z} | \exists \mathbf{x} M_{\langle \mathrm{NF},\mathrm{NF}\rangle}(\mathbf{x}, \mathbf{z})\} = \{\mathbf{z} | z_1 = z_2^2\}$$

or equivalently

$$\{\mathbf{z} | \neg\exists \mathbf{x} M_{\langle \mathrm{NF},\mathrm{NF}\rangle}(\mathbf{x}, \mathbf{z})\} = \{\mathbf{z} | z_1 - z_2^2 \neq 0\} \tag{2.19}$$

To fulfill expression (2.17) requires that there is a test, call it δ_1, such that $T_1(\mathbf{z}) \in \mathcal{R}_1$. This could be done choosing $T_1 = z_1 - z_2^2$, $\mathcal{R}_1^C = \{0\}$, and $\Phi_1 = \langle \mathrm{NF}, \mathrm{NF}\rangle$. Now (2.17) holds particularly for the system behavioral-mode $\langle \mathrm{NF}, \mathrm{NF}\rangle$.

The next system behavioral-mode in (2.18) is $\langle \mathrm{NF}, \mathrm{UF}\rangle$. Its model is $M_{\langle \mathrm{NF},\mathrm{UF}\rangle} = \{e_1, e_2\}$. The corresponding set is

$$\{\mathbf{z} | \neg\exists \mathbf{x} M_{\langle \mathrm{NF},\mathrm{UF}\rangle}(\mathbf{x}, \mathbf{z})\} = \{\mathbf{z} | z_1 < 0\} \tag{2.20}$$

A new test could be defined as $T_2 = z_1$, $\mathcal{R}_2 = \mathbb{R}_-$, and $\Phi_2 = \phi(s_1 = \mathrm{NF})$. The investigation of the remaining two system behavioral-modes in (2.18) implies that non of its corresponding model can be invalidated. For example is $M_{\langle \mathrm{UF},\mathrm{NF}\rangle} = \{e_2, e_3\}$. Since $x_2 \in \mathbb{R}$ then $z_2 \in \mathbb{R}$. Hence there is no $\mathbf{z}$ that invalidate $M_{\langle \mathrm{UF},\mathrm{NF}\rangle}$. No more tests in a diagnostic system are needed to fulfill (2.17). The sound and complete diagnostic system designed is shown in Table 2.4.

In the example the set $\{\mathbf{z} | \exists \mathbf{x} M_b(\mathbf{x}, \mathbf{z})\}$ is calculated for each system behavioral-mode $b \in \mathcal{B}$. This set is the orthogonal projection of the model variables $\mathbf{x}$ and

Table 2.4 *A complete and sound diagnostic system for the model in Table 2.3.*

test	$H_i^0 : sys \in \Phi_i$	M_i	T_i	$\mathcal{R}_i$
δ_1	$\phi(s_1 = NF \wedge s_2 = NF)$	$\{e_1, e_2, e_3\}$	$z_1 - z_2^2$	$\mathbb{R}\backslash\{0\}$
δ_2	$\phi(s_1 = NF)$	$\{e_1, e_2\}$	z_1	$\mathbb{R}_-$

$\mathbf{z}$ into $\mathbf{z}$. This is computationally complex and no automatic method exists in the general case.

To reduce the work of finding and designing tests, abstractions of the diagnostic models can be used to suggest suitable models, sets of equations, that can be used to derive tests. These models are smaller than the behavioral models and is therefore often easier to analytically analyze. One type of model abstraction is to use only the structure of the model and collect it in a *structural model*. In the next chapter structural models and there application to diagnosis is presented.

3

Structural Models and Their Properties

In the previous chapter we explained that the design of diagnostic systems can be divided into two steps. In the first step a set of models is suggested to be tested such that a diagnostic system based on these tests becomes sound and complete. In the second step a diagnostic test is designed for each suggested model to obtain a diagnostic system. As said in the previous chapter it is possible to suggest a set of models to design tests with less detailed knowledge than using a complete diagnostic model. One type of less detailed model that will be used in this thesis is a so called *structural model*. In Section 3.1, structural diagnostic model and the structure of a model is defined and explained. Furthermore two common representations of structural models are presented. In Section 3.2 a couple of scenarios are described when structural models can be helpful to use together with structural analysis in order to design a diagnostic system. In Section 3.3 we briefly discuss how structural models can be obtained. To solve the tasks stated in Section 3.2 fundamental structural properties are needed. These properties are presented in Section 3.4. Finally some basic results concerning these structural properties are given in Section 3.5.

3.1 Structural Models

The basic idea is that a structural model (Cassar & Staroswiecki 1997) contains information of which variables that are contained in each equation. For example, if e is an equation, f is a function, z and x are variables such that $e : f(x, z) = 0$, then the structure of this equation contains the knowledge that x and z are included in e, but nothing about the analytical expression of f.

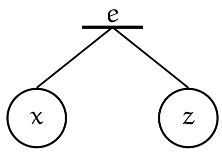

Figure 3.1 *Equation* $e : f(x, z) = 0$ *represented as a bipartite graph.*

A structural model will contain two different kinds of variables: known variables Z, e.g. sensor signals and actuators, and unknown variables X_u, for example internal states of the system and fault parameters. A structural model can be represented by an *incidence matrix* (Harary 1969, Carpanzano & Maffezzoni 1998). The rows correspond to equations and the columns to variables. A cross in position (i, j) tells that variable j is included in equation i. The equation e can then be written as

equation	unknown	known
	x	z
e	X	X

Before we continue with the next type of representation for structural models a useful operator is defined. If X is a set of variables and E is a set of equations, then the set of variables in X that is included in some equation in E is denoted $\text{var}_X E$. With abuse of notation, we will also allow equations as argument, e.g. $\text{var}_X e = \text{var}_X\{e\}$. A common representation of the structure of a model is to use a *bipartite graph*, which is a graph with edges Γ where the vertices of the graph can be partitioned into two sets E and X such that no two vertices in the same set have an edge in common. If E is a set of equations, X a set of variables, and Γ is a set of edges such that there is an edge in Γ between $x \in X$ and $e \in E$ if and only if $x \in \text{var}_X e$. Then the bipartite graph defined by E and X is denoted $\mathcal{G}(E, X)$. The equation e will be represented with a bipartite graph as shown in Figure 3.1. The equations are denoted with a line and variables are denoted with a circle.

3.1.1 The Structure of Dynamic Models

In this thesis mainly two different types of structural models are used to describe dynamic models. The difference between the two types of dynamic models is that the connections between variables and its derivatives are treated differently. The two types of structural models are in this thesis called *differentiated-separated structural-model* (DSSM) and *differentiated-lumped structural-model* (DLSM). An example will be used to show the differences. Consider the linear-dynamic model

$$\begin{aligned} e_1 &: \quad \dot{x}_1 = x_1 + u \\ e_2 &: \quad \dot{x}_1 = 2\,\dot{x}_2 \\ e_3 &: \quad 3\,x_1 - x_2 + y = 0 \\ \dot{e}_3 &: \quad 3\,\dot{x}_1 - \dot{x}_2 + \dot{y} = 0 \end{aligned} \tag{3.1}$$

where the first time-derivative of a variable is denoted with a dot and the an equation with a dot denotes an equation obtained by differentiating once. The DSSM is the structural model containing the most information. All equations, differentiated equations, variables, and differentiated variables are included in the DSSM. The DSSM of (3.1) is

equation	unknown				known		
	x_1	$\dot{x}_1$	x_2	$\dot{x}_2$	u	y	$\dot{y}$
e_1	X	X			X		
e_2		X		X			
e_3	X		X			X	
$\dot{e}_3$		X		X			X

The differentiated-lumped structural-model (DLSM) of (3.1) is

equation	unknown		known	
	x_1	x_2	u	y
e_1	X		X	
e_2	X	X		
e_3	X	X		X

Note that only non-differentiated equations e_1, e_2, and e_3 are included in the DLSM. In DLSM they are considered to be differential equations. The information of the differentiated equation $\dot{e}_3$ is therefore included in e_3. In DLSM each variable does not represent a value but a functions of time instead. Therefore only non-differentiated functions of time is included in the model, e.g. x_1 and x_2, but $\dot{x}_1$ is not included in the DLSM.

3.1.2 The Structure of Diagnostic Models and Diagnostic Systems

A *structural diagnostic model* is obtained if the analytical expressions of a diagnostic model are replaced by theirs structure. For example the structural diagnostic model of the diagnostic model in Table 2.3 is shown in Table 3.1. A diagnostic system for the diagnostic model in Table 2.3 is shown in Table 2.4. The structure of the diagnostic system in Table 2.4 is shown in Table 3.2. As can be seen in Table 3.2 there are three typical structural properties of a diagnostic test, i.e. the set defining the null hypothesis Φ_i, the set of equations M_i, and the set of known variables Z_i involved in the test.

Looking at Table 3.1 and Table 3.2 it is clear that for a given set of equations M_i both the other structural properties Φ_i and Z_i of a diagnostic system are easily computed by using only the structural diagnostic model. Before showing how the structural properties are computed, useful notation is introduced.

To simplify the notation we extend the use of operator ass to a set of equations M according to

$$\text{ass}\, M := \bigcap_{e \in M} \text{ass}\, e \tag{3.2}$$

Table 3.1 *The structural diagnostic model of the diagnostic model shown in Table 2.3.*

component	assumption	equation	unknown		known	
			x_1	x_2	z_1	z_2
Sensor 1	$\phi(s_1 = \mathrm{NF})$	e_1	X		X	
Comp	$\mathcal{B}$	e_2	X	X		
Sensor 2	$\phi(s_2 = \mathrm{NF})$	e_3		X		X

Table 3.2 *The structural interpretation of the complete and sound diagnostic system in Table 2.4 with respect to the diagnostic model in Table 2.3.*

test	M_i	Φ_i	Z_i
δ_1	$\{e_1, e_2, e_3\}$	$\phi(s_1 = \mathrm{NF} \wedge s_2 = \mathrm{NF})$	$\{z_1, z_2\}$
δ_2	$\{e_1, e_2\}$	$\phi(s_1 = \mathrm{NF})$	$\{z_1\}$

Then it follows for a given M_i that $\Phi_i = \text{ass}\, M_i$ and $Z_i = \text{var}_Z M_i$. Next an example is presented to show how the structure of a diagnostic system is obtained using only a structural diagnostic model.

Example 3.1 Continuation of Example 2.6. The results of extracting the structural information from the model in Table 2.3 and the system in Table 2.4 are shown in Table 3.1 and Table 3.2 respectively. Consider the test δ_1 in Table 3.2. Assume that $M_1 = \{e_1, e_2, e_3\}$ is given. Then by using the structural model in Table 3.1 it follows that

$$\Phi_1 = \text{ass}\, M_1 = \text{ass}\, e_1 \cap \text{ass}\, e_2 \cup \text{ass}\, e_3 = \phi(s_1 = \mathrm{NF} \wedge s_2 = \mathrm{NF}) \tag{3.3}$$

The known variables are calculated as

$$Z_1 = \text{var}_Z M_1 = \{z_1, z_2\} \tag{3.4}$$

The conclusion of the example is that given a set M_i it is straightforward to use the structural model to calculate Φ_i and Z_i. To design a diagnostic system the models M and their corresponding test quantities and a rejection regions have to be computed. A large part of this thesis is devoted to how to compute a set of models using mostly structural information. The design of a diagnostic test will not be a key issue. To read more about how diagnostic test are designed using the analytical properties of M, see for example (Frisk & Nyberg 2001, Nikoukhah 1998, Basseville & Nikiforov 1993). In the next section we the objectives of the structural analysis are presented. The objectives will imply the requirements of the set of models M.

3.2 Objectives of Structural Analysis

3.2.1 Pre-study without Analytical Models Available

When a diagnostic system is to be constructed for a system and its model, it is common that the diagnostic system is designed when the design of the system to be diagnosed is almost finished. A late start of the design of a diagnostic system is often due to the fact that diagnostic systems require detailed system knowledge. However, starting the design of the diagnostic system late can be costly. The chosen design of the system to be diagnosed can turn out to be a bad choice for diagnostic purpose. This means that the system together with the diagnostic system will either be less reliable than necessary or some modifications of the system to be diagnosed has to be done in order to improve the isolability. Adding extra sensors is one example of modification.

To find problems of different design solutions early in the design process, it is recommended that the design of the diagnostic system is going on during the design of the system to be diagnosed. As said earlier, the design of diagnostic systems requires a lot of detailed system knowledge, usually described in a model. Therefore it is not possible to do the complete design, before a model of the system is obtained. However, the structure of a model is much more easily obtained. To get the structure no analytical properties of the constraints have to be known, for example parameter values are not needed. Since the structure of a model can be obtained earlier in the design stage of the system to be diagnosed than an analytical model, the structure can be used to analyze the isolability earlier and in this way consider isolability aspects of design choices earlier.

During the design of the system and the diagnostic system, structural analysis can be used to find out the isolability limitations of the suggested design. Moreover, if a desired isolability of a diagnostic model cannot be obtained without for example adding sensors or making fault models, structural analysis can calculate which sensors or fault models that can be added to obtain the desired isolability. (Travé-Massuyès et al. 2001)

3.2.2 Deriving a Diagnostic System

Later in the design process, when the analytical relationships of the constraints are known, structural analysis can be used to find a set of models to be tested in order to design a good or even a sound and complete diagnostic system.

3.3 Deriving Structural Models

Deriving structural models can be done in different ways depending on the information available about the system to be diagnosed. If a structural model is needed to do an early isolability analysis, little information is available about the system to be diagnosed. Then a structural model can be obtained by using physical insights about which variables that have physical constraints to fulfill. If the system to be diagnosed consists of several components of the same or similar type the structural model for one such component can be used for all these components. An example will show how a structural model can be designed by using physical insights and without knowing analytical expressions.

Example 3.2 Consider the system in Example 2.1 that is shown in Figure 2.1. We know that the component behavioral modes that need to be considered are those shown in Table 2.1. First the mass-flow of water is transferred between the components. A correct working pump will control the flow of water into the tank. Using the same variable names as in Table 2.2 the $p = NF$ constraint will look like $c_1(u, f_{in})$. If the pump is stuck $p = PS$, then no water will be pumped into the tank, i.e. structurally there is a constraint $c_2(f_{in})$. Continuing in this way a structural diagnostic model can be obtained.

3.3.1 Given an Analytical Diagnostic Model

If an analytical model of the system to be diagnosed is known as for example in Table 2.2, the structural model is easily obtained by finding all including variables in each constraint. Note that parameter values need not to be known to obtain a structural model.

3.4 Structural Properties

To prestudy the isolability and to find which models to check for consistency are the two tasks that will be solved using structural analysis, as said in the previous section. Both these tasks are solved by finding which models that are rejectable. To design a diagnostic system it is also important to know which models that are rejectable models, because it is only rejectable models that can be used to find inconsistencies. For the isolability analysis, assume that all rejectable models of a diagnostic model is known and that exactly the same rejectable models are implied by two different system behavioral modes, then it is clear that these two modes can not be isolated from each other. This is a simple principle that can be used to find the isolability once the rejectable models are computed. Before some structural properties are presented it is useful to review some graph theory. The graph theory is a tool to explain and understand the structural properties that will be presented.

3.4.1 Some Basic Graph Theoretic Concepts

Structural models can, as mentioned earlier, be represented also as bipartite graphs (Carpanzano & Maffezzoni 1998, Cassar & Staroswiecki 1997). Given a graph with edges Γ a *matching* is a set of edges $\Gamma_0 \subseteq \Gamma$ such that no two edges have a vertex in common. A matching Γ_0 is a *maximal matching* if $|\Gamma_1| > |\Gamma_0|$ implies that Γ_1 is not a matching. Given a bipartite graph $\mathcal{G} = \mathcal{G}(E, X)$, a *complete matching* of E into X is a matching such that all vertices in E is an endpoint of an edge. A matching in $\mathcal{G}$ can equally well be a complete matching of X into E. A matching in $\mathcal{G}$ that firstly is a complete matching of E into X and secondly is a complete matching of X into E is a *perfect matching* in $\mathcal{G}$ (Grimaldi 1994).

Consider a matching Γ_0 in a graph with edges Γ. An *alternating path* is defined as a path whose edges are alternately in Γ_0 and in $\Gamma \backslash \Gamma_0$. An alternating path in a matching Γ_0 is an *augmented path* in Γ_0 if it begins and ends at two distinct unmatched vertices. For more details concerning bipartite graphs see (Asratian, Denley & Häggkvist 1998).

3.4.2 Structurally Overdetermined

There is a special and important type of rejectable models that are particular easy to characterize by using structural properties. They are models, where all of their unknown variables can be calculated or eliminated and then substituted into at least one remaining equation. This is sometimes called an *overdetermined model.* If no cancellations of variables occur when eliminating unknown variables, overdetermined models have a structural property, that will be defined next.

Definition 3.1 (Structurally Overdetermined). *An equation set E is said to be* ***structurally overdetermined*** *with respect to the set of variables X iff*

$$\forall X' \subseteq X, X' \neq \varnothing : |X'| < |\text{equ}_E(X')| \tag{3.5}$$

If an equation set H is said to be overdetermined, it means that the set of variables implicitly are defined as the unknown variables, i.e.

$$\forall X' \subseteq \text{var}_{X_u} H, X' \neq \varnothing : |X'| < |\text{equ}_H(X')| \tag{3.6}$$

In (Dulmage & Mendelsohn 1958) it is proven that there is a unique structurally overdetermined part of a model. A powerful way to obtain the structurally overdetermined part is to do a *canonical decomposition* (Dulmage & Mendelsohn 1958). The decomposition divides a model M in three parts: one structurally overdetermined denoted M^+, one structurally just-determined M^0, and one structurally underdetermined part M^-, see Figure 3.2. This is accomplished by first finding a maximal matching in the bipartite graph $\mathcal{G}(M, \text{var}_{X_u} M)$. Denote the assigned equations and variables in the maximal matching with M_m and X_m respectively. Now, the set of all equation vertices such that there is an alternating path from $M \backslash M_m$ is the structurally overdetermined part of the model M^+. The structurally underdetermined part of the model M^- is the set of equation vertices such that there is an alternating path from $\text{var}_{X_u} M \backslash X_m$. The remaining part of the model is the structurally just-determined part M^0. This decomposition can be obtained by using the command `dmperm` in matlab.

Example 3.3 Consider the diagnostic model in Table 2.3, the corresponding structural model in Table 3.1 and the diagnostic system in Table 2.4. The canonical decomposition of $M_1 = \{e_1, e_2, e_3\}$ is $M_1^+ = M_1$. Hence M_1 is structurally overdetermined and it is likely that a test can be obtained. Analytically this test

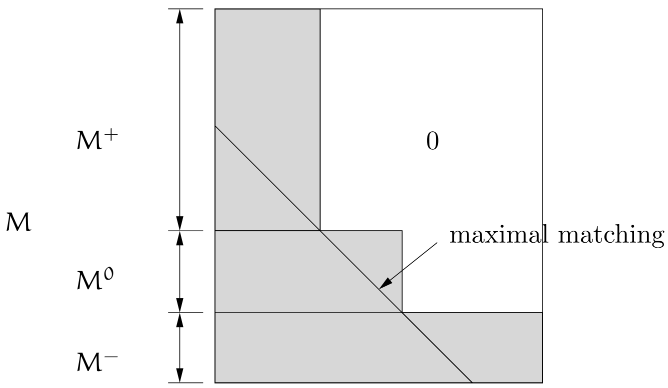

Figure 3.2 *Schematic illustration of a canonical decomposition of a model* M*. The rectangle represents an incident matrix of a model* M*. All entries in the white area are 0. In the gray area the entries are 0 or* X *and in the diagonal, which is a maximal matching, all entries are* X*.*

can be chosen as in Table 2.4. For $M_2 = \{e_1, e_2\}$ in Table 3.2, the canonical decomposition is $M_2^0 = M_2$, i.e. the entire model is structurally just-determined. Analytically it is possible to calculate the unknowns but it is not possible to eliminate the unknowns. This is true since $x_1 := z_1$ and $x_2 := \pm\sqrt{z_1}$. However, even if it is not possible to eliminate all unknown variables, a test can be designed, as seen in Table 2.4. It is not necessary to eliminate all unknown variables to design a test. It is sufficient that the range of the orthogonal projection of the valid model onto the space of observation does not equal the space of observation. To find all such sets, analytical properties are needed. For example $\{e_1, e_2\}$ has the same structure as $\{e_2, e_3\}$, but only $\{e_1, e_2\}$ can be used to derive a test.

3.4.3 Minimal Overdetermined Models

In the two-step approach for designing diagnostic systems, a test for each model found in the first step is to be designed in second step. In general it is easier to design tests for models with a small number of equations. Testing small models also has the advantage that each test becomes sensitive to few faults. This implies that small overdetermined models are especially interesting. One type of small overdetermined models are the minimal structurally overdetermined models which will be defined in the next two definitions, see (Pantelides 1988).

Definition 3.2 (Structurally Singular). *A finite set of equations* M *is* ***structurally singular*** *with respect to the set of variables* X *if* $|M| > |\mathrm{var}_X M|$.

Definition 3.3 (Minimal Structurally Singular). *A structurally singular set is a* ***minimal structurally singular*** *(MSS) set if none of its proper subsets are structurally singular.*

Given a model M the set of MSS sets in M is denoted mss$M$. Next some examples of the connections of other structural properties suggested by other authors are given. MSS is equivalent to structurally just-overdetermined or minimal structurally overdetermined models. A structurally just-overdetermined set is structurally overdetermined. A structurally overdetermined set is structurally singular. The next section is mainly for readers who want to read proofs, otherwise it is possible to directly start to read Chapter 4.

3.5 Structural Results

Before we present two characterizations of MSS sets a classical graph theoretical result is presented. The following theorem is often referred to as Hall's theorem (Harary 1969).

Theorem 3.1 (System of Distinct Representatives). *Let* $V = \{V_1, V_2, \cdots, V_m\}$ *be a set of objects and* $S = \{S_1, S_2, \cdots, S_n\}$ *a set of subsets of* V. *Then a complete matching of* S *into* V *exists iff* $\forall S' \subseteq S : |S'| \leq |\bigcup_{S_i \in S'} S_i|$.

Note that Theorem 3.1 can be used in two ways. The next two corollaries follow immediately from Theorem 3.1.

Corollary 3.2. *There is a complete matching of* E *into* X *iff* $\forall E' \subseteq E : |E'| \leq |\text{var}_X E'|$.

Corollary 3.3. *There is a complete matching of* X *into* E *iff* $\forall X' \subseteq X : |X'| \leq |\text{equ}_E(X')|$.

3.5.1 Characterizations of MSS Sets

In this section two equivalent characterizations of MSS sets will be presented and proven in the next two lemmas.

Lemma 3.4. *For all* $e \in E$ *there exists a perfect matching in* $(E \setminus \{e\}, \text{var}_{X_u} E)$ *if and only if* E *is an MSS set.*

Proof. $\Rightarrow$) From the hypothesis that there exist a perfect matching in $(E \setminus \{e\}, \text{var}_{X_u} E)$ for any $e \in E$, it follows that E is structurally singular, because $|\text{var}_{X_u} E| = |E \setminus \{e\}| < |E|$.

The set E is also minimal if no subset of E is structurally singular, i.e. $\forall \hat{E} \subset E : |\text{var}_{X_u} \hat{E}| \geq |\hat{E}|$. For each proper subset $\hat{E}$ of E, it is always possible to choose an equation e, such that $\hat{E} \subseteq E \setminus \{e\}$. Since there exists a perfect matching in $\mathcal{G}(E \setminus \{e\}, \text{var}_{X_u} E)$, according to the hypothesis, it follows that there exists a complete matching of $\hat{E}$ into $\text{var}_{X_u} E$.

Let $\mathsf{E} = \hat{\mathsf{E}}$ and $\mathsf{X} = \mathrm{var}_{\mathsf{X}_u}\mathsf{E}$ in Corollary 3.2, then

$$\forall \mathsf{E}' \subseteq \hat{\mathsf{E}} : |\mathsf{E}'| \leq |\mathrm{var}_{\mathrm{var}_{\mathsf{X}_u}\mathsf{E}}(\mathsf{E}')| = |\mathrm{var}_{\mathsf{X}_u}(\mathsf{E}')|. \tag{3.7}$$

Letting $\mathsf{E}' = \hat{\mathsf{E}}$, the inequality (3.7) becomes $|\hat{\mathsf{E}}| \leq |\mathrm{var}_{\mathsf{X}_u}(\hat{\mathsf{E}})|$. The conclusion is that $\hat{\mathsf{E}}$ is not structurally singular and since $\hat{\mathsf{E}}$ is an arbitrary chosen proper subset of E, it follows that E is an MSS set.

$\Leftarrow$) Take an arbitrary $e \in \mathsf{E}$ and let $\mathsf{E}' = \mathsf{E}\backslash\{e\}$. It is sufficient to prove that there exist a perfect matching in $\mathcal{G}(\mathsf{E}', \mathrm{var}_{\mathsf{X}}\mathsf{E})$. From the definition of MSS sets, it follows that $\forall \bar{\mathsf{E}} \subset \mathsf{E} : |\bar{\mathsf{E}}| \leq |\mathrm{var}_{\mathsf{X}}(\bar{\mathsf{E}})|$. Especially this is true for $\mathsf{E}' \subset \mathsf{E}$, i.e. $\forall \bar{\mathsf{E}} \subseteq \mathsf{E}' : |\bar{\mathsf{E}}| \leq |\mathrm{var}_{\mathsf{X}}(\bar{\mathsf{E}})|$. According to Corollary 3.2, there is a complete matching of E' into $\mathrm{var}_{\mathsf{X}}\mathsf{E}'$.

Since E is an MSS set and $\mathrm{var}_{\mathsf{X}}(\mathsf{E}') \subseteq \mathrm{var}_{\mathsf{X}}\mathsf{E}$ it follows that

$$|\mathsf{E}'| \leq |\mathrm{var}_{\mathsf{X}}(\mathsf{E}')| \leq |\mathrm{var}_{\mathsf{X}}\mathsf{E}| < |\mathsf{E}| = |\mathsf{E}'| + 1 \tag{3.8}$$

This implies that $|\mathsf{E}'| = |\mathrm{var}_{\mathsf{X}}\mathsf{E}'|$, hence the complete matching is a perfect matching in $\mathcal{G}(\mathsf{E}', \mathrm{var}_{\mathsf{X}}(\mathsf{E}'))$. The inequality (3.8) also implies that $|\mathrm{var}_{\mathsf{X}}(\mathsf{E}')| = |\mathrm{var}_{\mathsf{X}}\mathsf{E}|$, therefore $\mathrm{var}_{\mathsf{X}}(\mathsf{E}') = \mathrm{var}_{\mathsf{X}}\mathsf{E}$. The perfect matching in $\mathcal{G}(\mathsf{E}', \mathrm{var}_{\mathsf{X}}\mathsf{E}')$ is also a perfect matching in $\mathcal{G}(\mathsf{E}', \mathrm{var}_{\mathsf{X}}\mathsf{E})$. □

Lemma 3.5. *The set of equations* E *is an MSS set if and only if* E *is structurally overdetermined and* $|\mathsf{E}| = |\mathrm{var}_{\mathsf{X}_u}\mathsf{E}| + 1$.

Proof. Let X be the unknown variables $\mathsf{X} = \mathrm{var}_{\mathsf{X}_u}\mathsf{E}$. First we starts to prove that E is MSS implies that E is structurally overdetermined, i.e.

$$\forall \bar{\mathsf{X}} \subseteq \mathsf{X}, \bar{\mathsf{X}} \neq \varnothing : |\mathrm{equ}_{\mathsf{E}}(\bar{\mathsf{X}})| > |\bar{\mathsf{X}}| \tag{3.9}$$

Consider the negation of the conclusion. That is, E is an MSS set and

$$\exists \bar{\mathsf{X}} \subseteq \mathsf{X}, \bar{\mathsf{X}} \neq \varnothing : |\mathrm{equ}_{\mathsf{E}}(\bar{\mathsf{X}})| \leq |\bar{\mathsf{X}}|. \tag{3.10}$$

Let X' be an $\bar{\mathsf{X}}$ that fulfill (3.10). From Theorem 3.4 and from the fact that E is an MSS set, it follows that $\forall e \in \mathsf{E} : \mathcal{G}(\mathsf{E}\backslash\{e\}, \mathsf{X})$, contains a perfect matching. From the definition of perfect matching it particularly follows that there is a complete matching from X into $\mathsf{E}\backslash\{e\}$. The use of Corollary 3.3 makes it possible to write

$$\forall e \in \mathsf{E} \forall \bar{\mathsf{X}} \subseteq \mathsf{X} : |\bar{\mathsf{X}}| \leq |\mathrm{equ}_{\mathsf{E}\backslash\{e\}}(\bar{\mathsf{X}})|. \tag{3.11}$$

Since $\mathsf{X} = \mathrm{var}_{\mathsf{X}}\mathsf{E}$ it means that $\forall x \in \mathsf{X} \exists e \in \mathsf{E} : x \in \mathrm{var}_{\mathsf{X}_u} e$. Especially it holds that $\forall x \in \mathsf{X}' \exists e \in \mathsf{E} : x \in \mathrm{var}_{\mathsf{X}_u} e$ since $\varnothing \neq \mathsf{X}' \subseteq \mathsf{X}$. Hence $\mathrm{equ}_{\mathsf{E}}(\mathsf{X}') \neq \varnothing$. Now apply (3.11) to X' and an $e' \in \mathrm{equ}_{\mathsf{E}}(\mathsf{X}')$, that is

$$|\mathsf{X}'| \leq |\mathrm{equ}_{\mathsf{E}\backslash\{e'\}}(\mathsf{X}')|. \tag{3.12}$$

From $e' \in \mathrm{equ}_{\mathsf{E}}(\mathsf{X}')$ follows that $e' \in \mathrm{equ}_{e'}(\mathsf{X}')$, hence $|\mathrm{equ}_{\{e'\}}(\mathsf{X}')| = 1$. Adding $|\mathrm{equ}_{\{e'\}}(\mathsf{X}')| = 1$ on the right-hand side of (3.12), it becomes

$$\begin{aligned} |\mathsf{X}'| &< |\mathrm{equ}_{\mathsf{E}\backslash\{e'\}}(\mathsf{X}')| + \\ &+ |\mathrm{equ}_{\{e'\}}(\mathsf{X}')| = |\mathrm{equ}_{\mathsf{E}}(\mathsf{X}')|. \end{aligned} \tag{3.13}$$

This is a contradiction and it follows that E is structurally overdetermined. Now it remains to prove that $|E| = |X| + 1$. Since E is an MSS set it follows from Theorem 3.4 that there exists a perfect matching of $E\backslash\{e\}$ and X. Hence it follows that

$$|E| - 1 = |E\backslash\{e\}| = |X| \tag{3.14}$$

which complete the proof in the right direction.

Now, assume that $|E| = |X| + 1$ and E is structurally overdetermined. The statement E is structurally overdetermined can be written as (3.9). For an arbitrarily chosen $e \in E$ it follows from (3.9) that

$$\forall \bar{X} \subseteq X, \bar{X} \neq \varnothing : |\mathrm{equ}_{E\backslash\{e\}}(\bar{X})| \geq |\bar{X}| \tag{3.15}$$

Using 3.15 and Corollary 3.3 it follows that there is a complete matching of X into $E\backslash\{e\}$. Since $|E\backslash\{e\}| = |X|$ according to $|E| = |X| + 1$ it follows that the complete matching of X into $E\backslash\{e\}$ is a perfect matching. Since $e \in E$ was arbitrarily chosen it follows from Theorem 3.4 that E is an MSS set. □

4

Isolability Analysis of Diagnostic Systems

In this chapter the isolation capability of a diagnostic system is analyzed. A key property of diagnostic systems is the analytical isolability that is formally defined. Properties of isolability are given and representations are suggested.

In Section 4.1 the notion of structural isolability is defined. Structural isolability is needed to analyze the analytical isolability using less computational structural methods that only take structural models as input. It is proven that the structural isolability is a necessary condition for analytical isolability. Thereby it is possible to compute the structural isolability to get an analytical isolability limitation. In Section 4.2 an algorithm is presented that computes the structural isolability. Finally, in Section 4.7 desired isolability is defined that is an useful and intuitive way to express design specifications of diagnostic systems. A structural method is presented that takes the desired isolability together with a proposed diagnostic system as inputs and computes the missing isolability properties.

4.1 Structural and Analytical Isolation Capability

It is assumed that the structure of a diagnostic system Δ is given. The analysis uses only the structural information contained in Δ to decide which behavioral modes that pairwise can be isolated. To be able to state the problem formally two binary relations are defined.

Definition 4.1 ($\mathcal{I}_s^\Delta$, Structural Isolability of a Diagnostic System). *Given a diagnostic system Δ there is a binary relation $\mathcal{I}_s^\Delta$ on $\mathcal{B} \times \mathcal{B}$ defined as*

$$\mathcal{I}_s^\Delta = \{(b_1, b_2) | \exists \delta_i : (b_1 \notin \Phi_i \land b_2 \in \Phi_i)\} \tag{4.1}$$

The relation $\mathcal{I}_s^\Delta$ is called the ***structural isolability of a diagnostic system*** *Δ.*

If $(b_1, b_2) \in \mathcal{I}_s^\Delta$ we say that b_1 is *structurally isolable* from b_2 using Δ. Note that the definition of structural isolability of Δ assumes that the diagnostic system Δ is, or can be rewritten as, the particular type defined in Definition 2.3. However most on-line diagnostic systems are of this type. The idea behind Definition 4.1 is that if b_1 is structurally isolable from b_2 then there exists a test that can reject b_2 but not b_1. Note that only the structural properties of Δ are needed to calculate $\mathcal{I}_s^\Delta$.

Definition 4.2 ($\mathcal{I}^\Delta$, Analytical Isolability of a Diagnostic System). *Given a diagnostic system Δ there is a binary relation $\mathcal{I}^\Delta$ on $\mathcal{B} \times \mathcal{B}$ defined as*

$$\mathcal{I}^\Delta = \{(b_1, b_2) | \exists \mathbf{z} : \big(b_1 \in \mathcal{C}(\mathbf{z}) \wedge b_2 \notin \mathcal{C}(\mathbf{z})\big)\} \tag{4.2}$$

The relation $\mathcal{I}^\Delta$ is called the ***analytical isolability of the diagnostic system*** *Δ.*

If $(b_1, b_2) \in \mathcal{I}^\Delta$ we say that b_1 is *analytically isolable* from b_2 with the diagnostic system Δ. Definition 4.2 defines which behavioral modes that can be analytically isolable from each other. This property requires also the analytical properties of Δ. The isolation capability of a diagnostic system is limited to the analytical isolability and therefore the interesting relation is $\mathcal{I}^\Delta$. Sometimes it is difficult to calculate $\mathcal{I}^\Delta$ and sometimes only the structural properties of Δ are known. In both these cases it is still possible to calculate $\mathcal{I}_s^\Delta$. The structural isolability is a necessary condition for the analytical isolability as we show later. Since $\mathcal{I}_s^\Delta$ in both the mentioned situations can be calculated, properties of the analytical isolability are obtained. An example will show how the two definitions are applied to a diagnostic system.

Example 4.1 Consider the diagnostic system in Example 2.6, i.e.

Δ	$H_i^0 : \Phi_i$	M_i	T_i	$\mathcal{R}_i$
δ_1	$\{\langle NF, NF\rangle\}$	$\{e_1, e_2, e_3\}$	$z_1 - z_2^2$	$\mathbb{R}\backslash\{0\}$
δ_2	$\{\langle NF, UF\rangle\}$	$\{e_1, e_2\}$	z_1	$\mathbb{R}_-$

(4.3)

where

$$\mathcal{B} = \{\langle NF, NF\rangle, \langle NF, UF\rangle, \langle UF, NF\rangle, \langle UF, UF\rangle\} \tag{4.4}$$

Let this diagnostic system be denoted Δ. Now, we can determine for example if

$$(\langle UF, UF\rangle, \langle NF, NF\rangle) \in \mathcal{I}_s^\Delta \tag{4.5}$$

using expression (4.1). Since

$$\langle UF, UF\rangle \notin \Phi_1 \tag{4.6}$$

and

$$\langle NF, NF\rangle \in \Phi_1 \tag{4.7}$$

it follows that δ_1 together with (4.1) imply (4.5). Hence (4.5) is true, i.e. given the diagnostic system Δ $\langle UF, UF\rangle$ is structurally isolable from $\langle NF, NF\rangle$. In this example the diagnostic system Δ also implies that $\langle \mathsf{UF}, \mathsf{UF} \rangle$ is analytically isolable from $\langle \mathsf{NF}, \mathsf{NF} \rangle$ which will be explained next. Let $(z_1, z_2) = (5, 0)$ then

$$\mathsf{T}_1 = 5 \in \mathcal{R}_1 = \mathbb{R} \backslash \{0\} \tag{4.8}$$

and

$$\mathsf{T}_2 = 5 \notin \mathcal{R}_2 = \mathbb{R}_- \tag{4.9}$$

The diagnostic statement is then

$$\mathcal{C}(5, 0) = \mathcal{B} \cap \bigcap_{i:H_i^0 \text{ rejected}} \Phi_i^C = \Phi_1^C = \{\langle \mathsf{UF}, \mathsf{NF} \rangle, \langle \mathsf{NF}, \mathsf{UF} \rangle, \langle \mathsf{UF}, \mathsf{UF} \rangle\} \tag{4.10}$$

Hence $\langle \mathsf{UF}, \mathsf{UF} \rangle$ is a candidate because $\langle \mathsf{UF}, \mathsf{UF} \rangle \in \mathcal{C}(5, 0)$ and $\langle \mathsf{NF}, \mathsf{NF} \rangle$ is not a candidate because $\langle \mathsf{NF}, \mathsf{NF} \rangle \notin \mathcal{C}(5, 0)$. Definition 4.2 implies that

$$(\langle \mathsf{UF}, \mathsf{UF} \rangle, \langle \mathsf{NF}, \mathsf{NF} \rangle) \in \mathcal{I}^\Delta \tag{4.11}$$

In the previous example it turned out that the pair of behavioral modes was both structurally isolable and analytically isolable. The next theorem shows that the structural isolability is a necessary condition for the analytical isolability.

Theorem 4.1. *Given a diagnostic system Δ it holds that*

$$\mathcal{I}^\Delta \subseteq \mathcal{I}_s^\Delta \tag{4.12}$$

Proof. Take an arbitrary $(b_1, b_2) \in \mathcal{I}^\Delta$. From Definition 4.2 it follows that there exists a $\mathbf{z} = \mathbf{z}_0$ such that b_1 is a candidate and b_2 is not a candidate. From the definition of candidate it follows that

$$\mathcal{C}(\mathbf{z}_0) \tag{4.13}$$

and

$$b_2 \notin \mathcal{C}(\mathbf{z}_0) \tag{4.14}$$

The definition of the diagnostic statement gives

$$\mathcal{C}(\mathbf{z}_0) = \mathcal{B} \cap \bigcap_{i:H_i^0 \text{ rejected}} (\mathcal{B} \backslash \Phi_i) \tag{4.15}$$

From (4.14) and (4.15) it follows that

$$b_2 \notin \bigcap_{i:H_i^0 \text{ rejected}} (\mathcal{B} \backslash \Phi_i) \tag{4.16}$$

This means that there is a test δ_1 such that H_1^0 is rejected and

$$b_2 \notin (\mathcal{B} \backslash \Phi_1) \tag{4.17}$$

or equivalently

$$b_2 \in \Phi_1 \tag{4.18}$$

Since

$$b_1 \in \bigcap_{i:H_i^0 \text{ rejected}} (\mathcal{B} \setminus \Phi_i) \tag{4.19}$$

it means that

$$b_1 \in \mathcal{B} \setminus \Phi_i \tag{4.20}$$

or equivalently

$$b_1 \notin \Phi_i \tag{4.21}$$

for all δ_i such that H_i^0 is rejected. Hence it holds also for δ_1 and the theorem follows from (4.18) and (4.21). □

4.2 Algorithm Calculating Structural Isolation Capability

We can analyze the structure of a diagnostic system Δ to obtain $\mathcal{I}_s^\Delta$. According to Theorem 4.1, $\mathcal{I}_s^\Delta$ is a superset of $\mathcal{I}^\Delta$. Next an algorithm is presented that given a diagnostic system Δ calculates $\mathcal{I}_s^\Delta$.

Algorithm 4.1.
Input: $\mathcal{B}$ *and* Φ_i *of* Δ.

a) *Set* $\mathcal{I}_s^\Delta := \varnothing$.

b) *For each test* δ_i *set*

$$\mathcal{I}_s^\Delta := \mathcal{I}_s^\Delta \cup \{(b_1, b_2) | b_1 \notin \Phi_i \wedge b_2 \in \Phi_i\} \tag{4.22}$$

Output: $\mathcal{I}_s^\Delta$

The next theorem proves that the output of Algorithm 4.1 is the $\mathcal{I}_s^\Delta$ defined in Definition 4.1.

Theorem 4.2. *Algorithm 4.1 calculates* $\mathcal{I}_s^\Delta$.

To discriminate between $\mathcal{I}_s^\Delta$ in Definition 4.1 and $\mathcal{I}_s^\Delta$ found in the output of Algorithm 4.1, let the output be denoted $\mathcal{I}_{s,alg}^\Delta$ in the proof of Theorem 4.2.

Proof. Theorem 4.2 holds iff

$$\mathcal{I}_s^\Delta = \mathcal{I}_{s,alg}^\Delta \tag{4.23}$$

where $\mathcal{I}_s^\Delta$ is defined in (4.1) and $\mathcal{I}_{s,alg}^\Delta$ is defined as the output of Algorithm 4.1. The straightforward calculations proving the theorem are

$$\begin{aligned} \mathcal{I}_s^\Delta = & \ \{(b_1, b_2) | \exists \delta_i : \big(b_1 \notin \Phi_i \wedge b_2 \in \Phi_i\big)\} = \\ = & \ \{(b_1, b_2) | \textstyle\bigvee_{\delta_i} \big(b_1 \notin \Phi_i \wedge b_2 \in \Phi_i\big)\} = \\ = & \ \textstyle\bigcup_{\delta_i} \{(b_1, b_2) | b_1 \notin \Phi_i \wedge b_2 \in \Phi_i\} = \\ = & \ \mathcal{I}_{s,alg}^\Delta \end{aligned}$$

□

Next an example shows how the algorithm step by step works.

Example 4.2 Consider the diagnostic system (4.3). Apply Algorithm 4.1 to the structure of the diagnostic system (4.3). In step (a) $\mathcal{I}_s^\Delta := \varnothing$. From the diagnostic system it follows that

$$\begin{aligned}\Phi_1 &= \{\langle NF, NF\rangle\}\\ \Phi_2 &= \{\langle NF, UF\rangle\}\end{aligned} \tag{4.24}$$

When step (b) is applied to test 1 the conclusion is

$$\begin{aligned}\mathcal{I}_s^\Delta = &\ \{(b_1, b_2)|b_1 \in \mathcal{B}\backslash\{\langle NF, NF\rangle\} \wedge b_2 = \langle NF, NF\rangle\} =\\ = &\ \{(\langle UF, NF\rangle, \langle NF, NF\rangle), (\langle NF, UF\rangle, \langle NF, NF\rangle), (\langle UF, UF\rangle, \langle NF, NF\rangle)\}\end{aligned} \tag{4.25}$$

Test 2 implies that

$$\begin{aligned}\mathcal{I}_s^\Delta := &\ \mathcal{I}_s^\Delta \cup \{(b_1, b_2)|b_1 \in \mathcal{B}\backslash\{\langle NF, UF\rangle\} \wedge b_2 = \langle NF, UF\rangle\} =\\ = &\ \{(\langle UF, NF\rangle, \langle NF, NF\rangle), (\langle NF, UF\rangle, \langle NF, NF\rangle), (\langle UF, UF\rangle, \langle NF, NF\rangle),\\ &\ (\langle NF, NF\rangle, \langle NF, UF\rangle), (\langle UF, NF\rangle, \langle NF, UF\rangle), (\langle UF, UF\rangle, \langle NF, UF\rangle)\}\end{aligned} \tag{4.26}$$

Since there are only two tests, the output of Algorithm 4.1 is $\mathcal{I}_s^\Delta$ defined in (4.26).

4.3 Representing Isolability

As seen in the previous example the set representation of $\mathcal{I}_s^\Delta$ is difficult to interpret even for small size systems like (4.3). Another common representation of a *relation* $\mathcal{R}$ is to use a *relation (incidence) matrix* $R = (r_{ij})$. If $\mathcal{R}$ is a relation on a finite set $\mathcal{B}$ then

$$r_{ij} = \begin{cases} 1 \text{ if } (b_i, b_j) \in \mathcal{R} \\ 0 \text{ if } (b_i, b_j) \notin \mathcal{R} \end{cases} \tag{4.27}$$

Let the relation defined as complement set to $\mathcal{I}$ on $\mathcal{B}$ be denoted $\overline{\mathcal{I}}$.

Definition 4.3 (Structural (Analytical) Isolability Matrix). *Given $\mathcal{I}_s^\Delta$ ($\mathcal{I}^\Delta$) the **isolability matrix** of $\mathcal{I}_s^\Delta$ ($\mathcal{I}^\Delta$) is defined as the relation matrix to $\overline{\mathcal{I}_s^\Delta}$ ($\overline{\mathcal{I}^\Delta}$).*

The *structural (analytical) isolability matrix* is denoted I_s^Δ (I^Δ). To make the isolability matrix even easier to interpret, the ones are replaced with "X" and the zeros are left out. The interpretation of an "X" in position (i, j) is that for all different **z**, the diagnostic statement implies that b_j is a candidate if b_i is a candidate. Hence b_i is not analytically isolable from b_j. The structural isolability matrix I_s^Δ of $\mathcal{I}_s^\Delta$ in (4.26) is

present	necessary interpreted mode			
mode	$\langle NF, NF\rangle$	$\langle NF, UF\rangle$	$\langle UF, NF\rangle$	$\langle UF, UF\rangle$
$\langle NF, NF\rangle$	X		X	X
$\langle NF, UF\rangle$		X	X	X
$\langle UF, NF\rangle$			X	X
$\langle UF, UF\rangle$			X	X

(4.28)

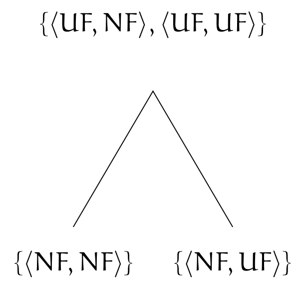

Figure 4.1 *The Hasse diagram of $\mathcal{P}_s^\Delta$ of (4.28).*

Note that the isolability matrix shows the complement set of the isolability relation. The isolability relation for (4.28) is the set corresponding to the blank entries in the isolability matrix.

It is interesting to note that $\overline{\mathcal{I}^\Delta}$ and $\overline{\mathcal{I}_s^\Delta}$ both are *reflexive* and *transitive*. It is possible to define a relation that also is *antisymmetric*. Relations with these three properties are *partial orders*.

Definition 4.4 ($\mathcal{P}_s^\Delta$ ($\mathcal{P}^\Delta$)). *Given a binary relation $\mathcal{I}_s^\Delta$ ($\mathcal{I}^\Delta$) on $\mathcal{B} \times \mathcal{B}$. Let $\mathcal{B}'$ be a partition of $\mathcal{B}$ defined as the set of the equivalent classes of $\overline{\mathcal{I}_s^\Delta}$ ($\overline{\mathcal{I}^\Delta}$) on $\mathcal{B}$. The equivalent class that contains b is denoted $[\mathsf{b}]$. The partial order $\mathcal{P}_s^\Delta$ ($\mathcal{P}^\Delta$) on $\mathcal{B}'$ is defined as*

$$([\mathsf{b}_1], [\mathsf{b}_2]) \in \mathcal{P}_s^\Delta \leftrightarrow (\mathsf{b}_1, \mathsf{b}_2) \in \overline{\mathcal{I}_s^\Delta} \tag{4.29}$$

The relation matrix of $\mathcal{P}_s^\Delta$ of (4.28) is

present modes	necessary interpreted modes {⟨NF, NF⟩}	{⟨NF, UF⟩}	{⟨UF, NF⟩, ⟨UF, UF⟩}
{⟨NF, NF⟩}	X		X
{⟨NF, UF⟩}		X	X
{⟨UF, NF⟩, ⟨UF, UF⟩}			X

(4.30)

The partial order $\mathcal{P}^\Delta$ has a nice interpretation that follows from Definition 4.2 and Definition 4.4 that is if b_1 is a candidate then all b_2 that fulfill

$$([\mathsf{b}_1], [\mathsf{b}_2]) \in \mathcal{P}^\Delta \tag{4.31}$$

are candidates.

A partial order $\mathcal{P}$ on a set $\mathcal{B}$ can be represented by a *Hasse diagram.* In such a diagram an element is represented with a node. If $([\mathsf{b}_1], [\mathsf{b}_2]) \in \mathcal{P}$ where $\mathsf{b}_1, \mathsf{b}_2 \in \mathcal{B}$, then b_1 is at a lower level then b_2, and there exists a path from b_1 upwards to b_2. The Hasse diagram for $\mathcal{P}_s^\Delta$ in (4.30) is shown in Figure 4.1. To explicitly state that a relation $\mathcal{R}$ is used to obtain the equivalent class $[\mathsf{b}]$, the notation $[\mathsf{b}]_\mathcal{R}$ will be used.

Corollary 4.3. *Given a diagnostic system* Δ *it holds that*

$$([b_1]_{\overline{\mathcal{I}_s^\Delta}}, [b_2]_{\overline{\mathcal{I}_s^\Delta}}) \in \mathcal{P}_s^\Delta \to ([b_1]_{\overline{\mathcal{I}^\Delta}}, [b_2]_{\overline{\mathcal{I}^\Delta}}) \in \mathcal{P}^\Delta \tag{4.32}$$

Proof. Take an arbitrary

$$([b_1]_{\overline{\mathcal{I}_s^\Delta}}, [b_2]_{\overline{\mathcal{I}_s^\Delta}}) \in \mathcal{P}_s^\Delta \tag{4.33}$$

From (4.29) and (4.33) it follows that

$$(b_1, b_2) \in \overline{\mathcal{I}_s^\Delta} \tag{4.34}$$

Using Theorem 4.1 and (4.34) it follows that

$$(b_1, b_2) \in \overline{\mathcal{I}^\Delta} \tag{4.35}$$

Finally (4.35) and (4.29) gives that

$$([b_1]_{\overline{\mathcal{I}^\Delta}}, [b_2]_{\overline{\mathcal{I}^\Delta}}) \in \mathcal{P}^\Delta \tag{4.36}$$

□

The interpretation of Corollary 4.3 is that if b_1 is a candidate then all b_2 that fulfill

$$([b_1], [b_2]) \in \mathcal{P}_s^\Delta \tag{4.37}$$

are candidates. In (4.31) we used the analytical properties of Δ to draw conclusions about the isolation capability but in (4.37) only the structural properties are used. From Figure 4.1 and Corollary 4.3 it is clear that e.g. if $\langle \mathsf{NF}, \mathsf{NF} \rangle$ is a candidate then $\langle \mathsf{UF}, \mathsf{NF} \rangle$ and $\langle \mathsf{UF}, \mathsf{UF} \rangle$ are candidates too.

4.4 Comparison between Structural and Analytical Isolability

From previous sections we know that structural isolability is a necessary condition for analytical isolability. In this section we use Example 4.2 to show the difference between the structural isolability and the analytical isolability. For the diagnostic system in Example 4.2, it is possible to calculate $\mathcal{I}^\Delta$. Before the calculations are carried out a useful definition is presented.

Definition 4.5 (Acceptance Set, $\mathcal{O}_{\delta_i}^\Delta$). *Given a diagnostic system* Δ *and one of its tests* δ_i *the* ***acceptance set*** *for* δ_i *is*

$$\mathcal{O}_{\delta_i}^\Delta := \{\mathbf{z} | \mathsf{T}_i(\mathbf{z}) \in \mathcal{R}_i^C\} \tag{4.38}$$

Theoretically $\mathcal{O}_{\delta_i}^\Delta$ contains equivalent information as T_i and $\mathcal{R}_i$. However in practice, T_i and $\mathcal{R}_i$ also express an efficient way to evaluate δ_i. In the next example

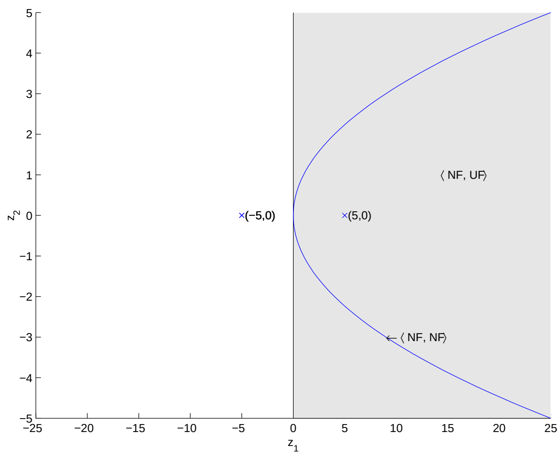

Figure 4.2 *The set $\mathcal{O}^{\Delta}_{\delta_1}$ and $\mathcal{O}^{\Delta}_{\delta_1}$ shown in the* **z***-plane*

$\mathcal{I}^{\Delta}$ is calculated for (4.3) and compared with $\mathcal{I}^{\Delta}_{s}$ which is calculated in Example 4.2.

Example 4.3 The continuation of Example 4.2. A good way to illustrate a static diagnostic system with two known variables is to plot the acceptance set of each test. For the diagnostic system (4.3) this plot is shown in Figure 4.2. The set $\mathcal{O}^{\Delta}_{\delta_1}$ is the parabola and $\mathcal{O}^{\Delta}_{\delta_2}$ is right half-plane. The set $\mathcal{O}^{\Delta}_{\delta_1}$ and $\mathcal{O}^{\Delta}_{\delta_2}$ divides the space of observations in 3 parts, i.e.

$$\mathbf{z} \in \mathcal{O}^{\Delta}_{\delta_1} \wedge \mathbf{z} \in \mathcal{O}^{\Delta}_{\delta_2} \tag{4.39}$$

or

$$\mathbf{z} \notin \mathcal{O}^{\Delta}_{\delta_1} \wedge \mathbf{z} \in \mathcal{O}^{\Delta}_{\delta_2} \tag{4.40}$$

or

$$\mathbf{z} \notin \mathcal{O}^{\Delta}_{\delta_1} \wedge \mathbf{z} \notin \mathcal{O}^{\Delta}_{\delta_2} \tag{4.41}$$

Each of these three cases implies a different set of candidates. If (4.39) holds then no null hypothesis is rejected and hence all behavioral modes are candidates. Since all behavioral modes is candidates no isolability property in $\mathcal{I}^{\Delta}$ is implied. If (4.40) holds then H^0_1 is rejected and the result is that all behavioral modes except for $\langle \mathsf{NF}, \mathsf{NF} \rangle$ are candidates. According to the definition of $\mathcal{I}^{\Delta}$ it follows that

$$\begin{aligned}&\{(b_1, b_2)|b_1 \in \mathcal{C}(\mathbf{z}) \wedge b_2 \notin \mathcal{C}(\mathbf{z})\} = \\ &\{(b_1, b_2)|b_1 \in \mathcal{B}\backslash\{\langle \mathsf{NF}, \mathsf{NF} \rangle\} \wedge b_2 = \langle \mathsf{NF}, \mathsf{NF} \rangle\} = \\ &\{(\langle \mathsf{UF}, \mathsf{NF} \rangle, \langle \mathsf{NF}, \mathsf{NF} \rangle), (\langle \mathsf{NF}, \mathsf{UF} \rangle, \langle \mathsf{NF}, \mathsf{NF} \rangle), (\langle \mathsf{UF}, \mathsf{UF} \rangle, \langle \mathsf{NF}, \mathsf{NF} \rangle)\} \in \mathcal{I}^{\Delta}\end{aligned} \tag{4.42}$$

Finally if (4.41) holds then both null hypothesis are rejected and the candidates are $\langle \mathsf{UF}, \mathsf{NF} \rangle$ and $\langle \mathsf{UF}, \mathsf{UF} \rangle$. This case implies that

$$\{(b_1, b_2)|b_1 \in \{\langle \mathsf{UF}, \mathsf{NF} \rangle, \langle \mathsf{UF}, \mathsf{UF} \rangle\} \wedge b_2 \in \{\langle \mathsf{NF}, \mathsf{NF} \rangle, \langle \mathsf{NF}, \mathsf{UF} \rangle\}\} \in \mathcal{I}^{\Delta} \tag{4.43}$$

$\{\langle \mathsf{UF}, \mathsf{NF}\rangle, \langle \mathsf{UF}, \mathsf{UF}\rangle\}$

|

$\{\langle \mathsf{NF}, \mathsf{UF}\rangle\}$

|

$\{\langle \mathsf{NF}, \mathsf{NF}\rangle\}$

Figure 4.3 *Hasse diagram of (4.45).*

It can be realized that $\mathcal{I}^\Delta$ is the union of the set in (4.42) and set in (4.43), i.e.

$$\begin{aligned} \mathcal{I}^\Delta = \; & \{(\langle \mathsf{UF}, \mathsf{NF}\rangle, \langle \mathsf{NF}, \mathsf{NF}\rangle), (\langle \mathsf{NF}, \mathsf{UF}\rangle, \langle \mathsf{NF}, \mathsf{NF}\rangle), (\langle \mathsf{UF}, \mathsf{UF}\rangle, \langle \mathsf{NF}, \mathsf{NF}\rangle), \\ & (\langle \mathsf{UF}, \mathsf{NF}\rangle, \langle \mathsf{NF}, \mathsf{UF}\rangle), (\langle \mathsf{UF}, \mathsf{UF}\rangle, \langle \mathsf{NF}, \mathsf{UF}\rangle)\} \end{aligned} \tag{4.44}$$

The analytical isolability matrix I^Δ is

present mode	necessary interpreted mode $\langle \mathsf{NF}, \mathsf{NF}\rangle$	$\langle \mathsf{NF}, \mathsf{UF}\rangle$	$\langle \mathsf{UF}, \mathsf{NF}\rangle$	$\langle \mathsf{UF}, \mathsf{UF}\rangle$
$\langle \mathsf{NF}, \mathsf{NF}\rangle$	X	**X**	X	X
$\langle \mathsf{NF}, \mathsf{UF}\rangle$		X	X	X
$\langle \mathsf{UF}, \mathsf{NF}\rangle$			X	X
$\langle \mathsf{UF}, \mathsf{UF}\rangle$			X	X

(4.45)

Comparing $\mathcal{I}_s^\Delta$ and $\mathcal{I}^\Delta$ reveals that

$$\mathcal{I}_s^\Delta = \mathcal{I}^\Delta \cup \{(\langle \mathsf{NF}, \mathsf{NF}\rangle, \langle \mathsf{NF}, \mathsf{UF}\rangle)\} \tag{4.46}$$

This difference is marked with a bold "**X**" in (4.45).

Note that in the previous example $\mathcal{I}^\Delta \subseteq \mathcal{I}_s^\Delta$ as stated in Theorem 4.1. Note also that $\mathcal{I}^\Delta \neq \mathcal{I}_s^\Delta$. The origin of the difference can be understood looking at (4.26) and (4.43). The same analytical isolability as the structural isolability in (4.26) is obtained, if in this case only the null hypothesis of test 2 is rejected. However,

$$\mathcal{O}_{\delta_1}^\Delta \subset \mathcal{O}_{\delta_2}^\Delta \tag{4.47}$$

implies that the null hypotheses of test 1 is rejected when the null hypothesis of test 2 is rejected. Hence the tests have analytical constraints of which sets of tests that can be invalidated. In the structural analysis these constraints are not known and therefore not considered. This implies that the structural isolability is more optimistic than the analytical isolability.

4.5 Improving Structural Isolability

In the previous example, it was shown that $\mathcal{I}_s^\Delta \neq \mathcal{I}^\Delta$. Next a method is described that makes the structural isolability less optimistic. The key issue is to change the structural representation of a diagnostic system without changing the analytical properties. Then the structural representation which gives the "best" structural isolability can be used. The structural property that is going to be changed is the null hypothesis. To be able to compare two different behavioral mode assumptions the notion *weaker* will be defined.

Definition 4.6 (Weaker). *If Φ_1 and $\Phi_2 \subset \Phi_1$ are two set of system behavioral-modes that define the null hypotheses of two tests respectively, then Φ_1 is **weaker** than Φ_2.*

A way to compare the analytical properties of two diagnostic systems is defined as follows.

Definition 4.7 ($\Delta = \bar{\Delta}$). *If Δ and $\bar{\Delta}$ are two diagnostic systems, $\mathcal{C}(\mathbf{z})$ and $\bar{\mathcal{C}}(\mathbf{z})$ are their diagnostic statement respectively, then the diagnostic systems are equal, i.e. $\Delta = \bar{\Delta}$, iff*

$$\forall \mathbf{z} : \mathcal{C}(\mathbf{z}) = \bar{\mathcal{C}}(\mathbf{z}) \tag{4.48}$$

Next a theorem is presented that describes how the structural representation of a diagnostic system can be changed, without affecting the analytical properties.

Theorem 4.4. *Let a diagnostic system Δ be given. Suppose that there are two test δ_1 and δ_2 with the following property*

$$\mathcal{O}^{\Delta}_{\delta_1} \subseteq \mathcal{O}^{\Delta}_{\delta_2} \tag{4.49}$$

If a new diagnostic system $\bar{\Delta}$ is defined identical to Δ except that Φ_2 is replaced with

$$\bar{\Phi}_2 = \Phi_1 \cup \Phi_2 \tag{4.50}$$

then $\Delta = \bar{\Delta}$.

If two diagnostic systems are identical then all their T_i, $\mathcal{R}_i$, and Φ_i respectively are exactly the same. Since Δ and $\bar{\Delta}$ in Theorem 4.4 are identical it follows that

$$\bar{\Phi}_i = \Phi_i \quad \text{for } i \neq 2 \tag{4.51}$$

Proof. According to (4.48), it is equivalent to show that $\Delta = \bar{\Delta}$ and

$$\forall \mathbf{z} : \mathcal{C}(\mathbf{z}) = \bar{\mathcal{C}}(\mathbf{z}) \tag{4.52}$$

For all $\mathbf{z} \in \mathcal{O}^{\Delta}_{\delta_2}$, it follows trivially that $\mathcal{C}(\mathbf{z}) = \bar{\mathcal{C}}(\mathbf{z})$. Consider any $\mathbf{z}$ such that

$$\mathbf{z} \notin \mathcal{O}^{\Delta}_{\delta_2} \tag{4.53}$$

Then from (4.49) it follows that

$$\mathbf{z} \notin \mathcal{O}^{\Delta}_{\delta_1} \tag{4.54}$$

From the definition of diagnostic statement it follows for Δ that

$$\mathcal{C}(\mathbf{z}) = \bigcap_{i:H^0_i \text{ rejected}} \Phi^C_i = A \cap \Phi^C_1 \cap \Phi^C_2 \tag{4.55}$$

for some set A. For $\bar{\Delta}$ it follows that

$$\bar{\mathcal{C}}(\mathbf{z}) = \bigcap_{i:H_i^0 \text{ rejected}} \bar{\Phi}_i^C = A \cap \Phi_1^C \cap \bar{\Phi}_2^C \tag{4.56}$$

Expression (4.50) can be rewritten as

$$\bar{\Phi}_2^C = \Phi_1^C \cap \Phi_2^C \tag{4.57}$$

From (4.55), (4.56) and (4.57) it follows that $\mathcal{C}(\mathbf{z}) = \bar{\mathcal{C}}(\mathbf{z})$. Since it holds that $\mathcal{C}(\mathbf{z}) = \bar{\mathcal{C}}(\mathbf{z})$ for all $\mathbf{z}$ it follows that $\Delta = \bar{\Delta}$. □

Theorem 4.4 shows that the behavioral mode assumptions can be made weaker with the method described in Theorem 4.4 without affecting the diagnostic statement.

Example 4.4 Continuation of Example 4.3. In (4.47) it was concluded that

$$\mathcal{O}_{\delta_1}^{\Delta} \subset \mathcal{O}_{\delta_2}^{\Delta} \tag{4.58}$$

Then condition (4.49) is fulfilled and Theorem 4.4 can be applied. A new diagnostic system Δ' is constructed where

$$\Phi_2' = \Phi_1 \cup \Phi_2 = \{\langle \mathsf{NF}, \mathsf{NF}\rangle, \langle \mathsf{NF}, \mathsf{UF}\rangle\} \tag{4.59}$$

Then $\Delta = \Delta'$, i.e. the two diagnostic systems always produces the same candidates. This can also be interpreted as two different representations of equal diagnostic systems.

Now we know that there are different descriptions of equal diagnostic systems. This is interesting because these different descriptions change only structural properties. That is equal diagnostic systems with the same analytical isolability relation can have different structural isolability relations. Hence there could be some descriptions that produce better structural isolability. The next theorem describes how to chose a representation of a diagnostic system such that better structural isolability can be obtained.

Theorem 4.5. *Let two diagnostic systems Δ and $\bar{\Delta}$ be defined as in Theorem 4.4. Then it holds that*

$$\mathcal{I}_s^{\bar{\Delta}} \subseteq \mathcal{I}_s^{\Delta} \tag{4.60}$$

Proof. Let any $(b_1, b_2) \in \mathcal{I}_s^{\bar{\Delta}}$ such that

$$b_1 \notin \bar{\Phi}_2 \wedge b_2 \in \bar{\Phi}_2 \tag{4.61}$$

It follows from (4.50) that

$$b_1 \notin \bar{\Phi}_2 \rightarrow \big((b_1 \notin \Phi_1) \wedge (b_1 \notin \Phi_2)\big) \tag{4.62}$$

and

$$b_2 \in \bar{\Phi}_2 \rightarrow \big((b_2 \in \Phi_1) \vee (b_2 \in \Phi_2)\big) \tag{4.63}$$

Hence according to (4.62) and (4.63), δ_1 or δ_2 secures that $(b_1, b_2) \in \mathcal{I}_s^{\Delta}$. □

Example 4.5 Let Algorithm 4.1 be applied to Δ'. Everything will be identical with Example 4.2 until Step (b) is applied to δ_2'. The corresponding expression to (4.26) is

$$\begin{aligned}\mathcal{I}_s^{\Delta'} := \; & \mathcal{I}_s^{\Delta'} \cup \{(b_1, b_2) | b_1 \in \{\langle UF, NF\rangle, \langle UF, UF\rangle\} \wedge b_2 \in \{\langle NF, NF\rangle, \langle NF, UF\rangle\} = \\ = \; & \{(\langle UF, NF\rangle, \langle NF, NF\rangle), (\langle NF, UF\rangle, \langle NF, NF\rangle), (\langle UF, UF\rangle, \langle NF, NF\rangle), \\ & (\langle UF, NF\rangle, \langle NF, UF\rangle), (\langle UF, UF\rangle, \langle NF, UF\rangle)\}\end{aligned} \tag{4.64}$$

A comparison between the different isolability relations gives that

$$\mathcal{I}^{\Delta} = \mathcal{I}_s^{\Delta'} \subset \mathcal{I}_s^{\Delta} \tag{4.65}$$

This example and Theorem 4.5 show that using weaker behavioral mode assumptions the structural isolability can be less optimistic, i.e. be more similar to the analytical isolability.

4.6 Optimality Condition for Structural Isolability

The previous example showed that improvements of the structural isolability can be made if the behavioral mode assumptions are made weaker. In Example 4.5, it holds that $\mathcal{I}_s^{\Delta'} = \mathcal{I}^{\Delta}$. Is there a condition, that is easy to check and that guarantees that the structural isolability is equal to the analytical isolability? The next example will show that it is not sufficient that all assumptions are made as strong as possible according to the method described in Theorem 4.4.

Example 4.6 Consider a diagnostic system Δ such that $\mathcal{B} = \{b_1, b_2, b_3\}$ and

Δ	$H_i^0 : sys \in \Phi_i$	$\mathcal{O}_{\delta_i}^{\Delta}$
δ_1	$\{b_1\}$	$\{0, 1\}$
δ_2	$\{b_1\}$	$\{0, 2\}$
δ_3	$\{b_2\}$	$\{0, 3\}$

(4.66)

No assumption in this model can be made weaker. The analytical isolability matrix I^{Δ} is

present mode	necessary interpreted mode b_1	b_2	b_3
b_1	X	X	X
b_2		X	X
b_3			X

(4.67)

and the structural isolability matrix I_s^Δ is

present mode	necessary interpreted mode b_1	b_2	b_3
b_1	X		X
b_2		X	X
b_3			X

(4.68)

Hence $\mathcal{I}^\Delta \neq \mathcal{I}_s^\Delta$.

Since there is no condition to the authors knowledge that is easy to check, a more complicated condition that guarantees that the structural isolability relation is equal to the analytical isolability relation is given in the next theorem.

Theorem 4.6. $\mathcal{I}_s^\Delta = \mathcal{I}^\Delta$ *if and only if*

$$\forall b_i, b_j \in \mathcal{B}(\forall \mathbf{z}(b_i \in \mathcal{C}(\mathbf{z}) \to b_j \in \mathcal{C}(\mathbf{z})) \to \forall \delta_k(b_j \in \Phi_k \to b_i \in \Phi_k)) \tag{4.69}$$

Proof. Let $b_1, b_2 \in \mathcal{B}$ be arbitrarily chosen. It holds that (4.69) is equivalent to

$$\forall \mathbf{z}(b_1 \notin \mathcal{C}(\mathbf{z}) \vee b_2 \in \mathcal{C}(\mathbf{z})) \to \forall \delta_k(b_2 \notin \Phi_k \vee b_1 \in \Phi_k) \tag{4.70}$$

or

$$\exists \delta_k(b_2 \in \Phi_k \wedge b_1 \notin \Phi_k) \to \exists \mathbf{z}(b_1 \in \mathcal{C}(\mathbf{z}) \wedge b_2 \notin \mathcal{C}(\mathbf{z})) \tag{4.71}$$

This can be written as

$$(b_1, b_2) \in \mathcal{I}_s^\Delta \to (b_1, b_2) \in \mathcal{I}^\Delta \tag{4.72}$$

Since b_1 and b_2 were arbitrarily chosen it follows that

$$\mathcal{I}_s^\Delta \subseteq \mathcal{I}^\Delta \tag{4.73}$$

Theorem 4.1 concludes the proof. □

4.7 Practical Use of Structural Analysis of (Analytical) Isolability

It is not uncommon that diagnostic systems are introduced to alarm if the diagnosed system is working in a behavioral mode that is dangerous for humans or can lead to serious damage. When the diagnostic system alarms suitable precautions can be taken to avoid accidents. If there are several different dangerous behavioral modes with completely different precautions, it becomes important to isolate which of these behavioral modes that are present so the right precaution can be taken. This means that certain behavioral modes are important to isolate. This knowledge can be described as the *desired (analytical) isolability*.

Definition 4.8 (Desired Isolability $\mathcal{I}_d$). *Given all system behavioral modes $\mathcal{B}$ there is a binary relation defined $\mathcal{I}_d$ on $\mathcal{B} \times \mathcal{B}$ that defines the* ***desired isolability****.*

There are two main reasons why we would like to analyze a diagnostic system structurally to find $\mathcal{I}_s^\Delta$. The first reason is that the analytical analysis is too complex to carry out or unnecessary. The second reason is, as we will see in the next chapter, that the diagnostic system is under construction. This means that the diagnostic system for the moment consists of a set of potential test. The analytical properties of the tests are not known. Then if the result of the structural analysis looks promising the analytical properties of the tests are derived. The isolability analysis presented in this chapter can be used to answer the following question: Given the structural properties of a diagnostic system Δ find if it is possible that $\mathcal{I}_d \subseteq \mathcal{I}^\Delta$. If it is not possible that $\mathcal{I}_d \subseteq \mathcal{I}^\Delta$ which properties cannot be obtained with Δ? According to Theorem 4.4 use the strongest possible assumptions to get the best results of the structural analysis. Apply Algorithm 4.1 to Δ and find $\mathcal{I}_s^\Delta$. According to Theorem 4.1 it holds that $\mathcal{I}^\Delta \subseteq \mathcal{I}_s^\Delta$. If it holds that

$$\mathcal{I}_d \subseteq \mathcal{I}_s^\Delta \tag{4.74}$$

then it is possible that

$$\mathcal{I}_d \subseteq \mathcal{I}^\Delta \tag{4.75}$$

Contrary if

$$\mathcal{I}_d \nsubseteq \mathcal{I}_s^\Delta \tag{4.76}$$

then the diagnostic system Δ has *not* the desired (analytical) isolation capability. Those isolability properties that are missing to make it possible to obtain the desired isolability can be found using

$$\mathcal{I}_d \backslash \mathcal{I}_s^\Delta \subseteq \mathcal{I}_d \backslash \mathcal{I}^\Delta \tag{4.77}$$

The superset in the previous expression is the isolability properties missing to obtain the desired isolability. Hence $\mathcal{I}_s^\Delta \backslash \mathcal{I}_d$ is some of the missing isolability properties. Finally an example will show how desired isolability is defined and how conclusions are drawn using structural analysis applied to a diagnostic system.

Example 4.7 Continuation of Example 4.5. Assume that the desired isolability is defined with the following matrix

present mode	necessary interpreted mode ⟨NF, NF⟩	⟨NF, UF⟩	⟨UF, NF⟩	⟨UF, UF⟩
⟨NF, NF⟩	X	X	X	X
⟨NF, UF⟩		X	X	X
⟨UF, NF⟩			X	X
⟨UF, UF⟩				X

(4.78)

Note that the desired isolability relation is defined as the complement of the corresponding set to the desired isolability relation matrix. From the analysis in Example 4.5 it was found that

$$\mathcal{I}_d = \mathcal{I}_s^\Delta \cup \{(\langle \mathsf{UF}, \mathsf{UF} \rangle, \langle \mathsf{UF}, \mathsf{NF} \rangle)\} \tag{4.79}$$

Hence Δ cannot fulfill the desired isolability. The missing isolability properties are found according to (4.77) as

$$\mathcal{I}_{\mathsf{d}} \backslash \mathcal{I}_{s}^{\Delta} = \{(\langle \mathsf{UF}, \mathsf{UF} \rangle, \langle \mathsf{UF}, \mathsf{NF} \rangle)\} \tag{4.80}$$

A remedy to this shortage can be found using the definition of $\mathcal{I}_{s}^{\Delta}$. Assume that $(\langle \mathsf{UF}, \mathsf{UF} \rangle, \langle \mathsf{UF}, \mathsf{NF} \rangle) \in \mathcal{I}_{s}^{\Delta}$ then it follows that there is a test such that $\langle \mathsf{UF}, \mathsf{NF} \rangle \in \Phi$ and $\langle \mathsf{UF}, \mathsf{UF} \rangle \notin \Phi$. Since this diagnostic system is sound as shown in Example 2.6, there is no more information to obtain from the diagnostic model. Hence a more detailed model is needed to obtain the desired isolability.

5

Analytical Characterization of Sound and Complete Diagnostic Systems

Tests are designed by the use of different models, i.e. subsets of equations of the diagnostic model. In this chapter we investigate for which models that it is important to design tests, in order to get different desired properties of the diagnostic system, e.g. completeness and soundness. A key result is a necessary and sufficient condition for which sets of models that must be used to design a sound and complete diagnostic system. Using this result it is possible to calculate a set of minimum number of models that corresponds to a sound and complete diagnostic system.

5.1 Complete Diagnostic System

Completeness of a diagnostic system is independent of which models that are checked and how many tests that are used. For completeness the important property of the diagnostic system is how each test δ_i is designed, i.e. how Φ_i and $\mathcal{O}_{\delta_i}$ are chosen in relation to the diagnostic model. The next theorem states how the tests must be designed to produce a complete diagnostic system. Before the theorem is presented some useful notation is introduced. Let $\mathbb{M}$ be a diagnostic model and $\mathcal{B}$ its system behavioral modes. The expression $M \subseteq^* \mathbb{M}$ denotes that $M \subseteq M_b$ for some $b \in \mathcal{B}$. Moreover if $\mathbb{M}$ is a diagnostic model and $M \subseteq^* \mathbb{M}$ then

$$\mathcal{O}_M^{\mathbb{M}} := \{\mathbf{z} | \exists \mathbf{x} : M(\mathbf{x}, \mathbf{z})\} \tag{5.1}$$

Theorem 5.1 (Complete Diagnostic System). *Given a diagnostic model* $\mathbb{M}$ *and a diagnostic system* Δ*,* Δ *is complete with respect to* $\mathbb{M}$ *if and only if all tests*

δ_i *are designed such that*

$$\bigcup_{b \in \Phi_i} \mathcal{O}^{\mathbb{M}}_{M_b} \subseteq \mathcal{O}^{\Delta}_{\delta_i} \tag{5.2}$$

where Φ_i *is the set of behavioral modes that corresponds to* δ_i.

Before we prove Theorem 5.1 some useful formulas are summarized. The diagnostic statement is calculated as

$$\mathcal{C}(\mathbf{z}) = \mathcal{B} \cap \bigcap_{i:H^0_i \text{ rejected}} \Phi^C_i = \mathcal{B} \cap \bigcap_{i:\mathbf{z} \notin \mathcal{O}_{\delta_i}} \Phi^C_i \tag{5.3}$$

For a complete diagnostic system it holds that

$$\forall \mathbf{z}(\mathcal{D}(\mathbf{z}) \subseteq \mathcal{C}(\mathbf{z})) \tag{5.4}$$

Finally it holds that

$$b \in \mathcal{D}(\mathbf{z}) \leftrightarrow \mathbf{z} \in \mathcal{O}^{\mathbb{M}}_{M_b} \tag{5.5}$$

Proof. We start to show that the tests in a complete Δ satisfy (5.2). Take an arbitrary δ_i in Δ. We will show that

$$\forall \mathbf{z}(\mathbf{z} \notin \mathcal{O}^{\Delta}_{\delta_i} \rightarrow \mathbf{z} \notin \bigcup_{b \in \Phi_i} \mathcal{O}^{\mathbb{M}}_{M_b}) \tag{5.6}$$

which is equivalent to (5.2). Take an arbitrary $\mathbf{z} = \mathbf{z}_0$ such that

$$\mathbf{z}_0 \notin \mathcal{O}^{\Delta}_{\delta_i} \tag{5.7}$$

From (5.3) and (5.7) it follows that

$$\mathcal{C}(\mathbf{z}_0) \subseteq \Phi^C_i \tag{5.8}$$

According to (5.4) and (5.8) it holds that

$$\mathcal{D}(\mathbf{z}_0) \subseteq \Phi^C_i \tag{5.9}$$

Expression (5.9) implies that

$$\forall b \in \Phi_i : b \notin \mathcal{D}(\mathbf{z}_0) \tag{5.10}$$

Using (5.5) and (5.10) implies that

$$\forall b \in \Phi_i : \mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M_b} \tag{5.11}$$

But then

$$\mathbf{z}_0 \notin \bigcup_{b \in \Phi_i} \mathcal{O}^{\mathbb{M}}_{M_b} \tag{5.12}$$

Since $\mathbf{z}_0$ was arbitrarily chosen expression (5.6) follows from (5.7) and (5.12).

Now it remains to prove that Δ is complete if (5.2) holds for all δ_i. Assume that Δ is such that all tests δ_i fulfill (5.2). We will show that Δ is complete, i.e. Δ has the property expressed in (5.4). Take an arbitrarily chosen $\mathbf{z} = \mathbf{z}_0$ and an arbitrary b_0 such that

$$b_0 \in \mathcal{D}(\mathbf{z}_0) \tag{5.13}$$

or equivalently

$$\mathbf{z}_0 \in \mathcal{O}^{\mathbb{M}}_{M_{b_0}} \tag{5.14}$$

Let a set A be defined such that

$$i \in A \leftrightarrow b_0 \in \Phi_i \tag{5.15}$$

Then it holds that

$$\forall i \in A : \mathcal{O}^{\mathbb{M}}_{M_{b_0}} \subseteq \bigcup_{b \in \Phi_i} \mathcal{O}^{\mathbb{M}}_{M_b} \tag{5.16}$$

From (5.14) and (5.16) it follows that

$$\forall i \in A : \mathbf{z}_0 \in \bigcup_{b \in \Phi_i} \mathcal{O}^{\mathbb{M}}_{M_b} \tag{5.17}$$

The condition (5.2) gives that

$$\forall i \in A : \mathbf{z}_0 \in \mathcal{O}^{\Delta}_{\delta_i} \tag{5.18}$$

According to (5.3) and (5.18) it holds that

$$\mathcal{C}(\mathbf{z}_0) = \mathcal{B} \cap \bigcap_{\mathbf{z}_0 \notin \mathcal{O}^{\Delta}_{\delta_i} \wedge i \notin A} \Phi_i^C \supseteq \bigcap_{i \notin A} \Phi_i^C \tag{5.19}$$

Expression (5.15) can be rewritten as

$$i \notin A \leftrightarrow b_0 \in \Phi_i^C \tag{5.20}$$

From (5.19) and (5.20) it follows that

$$b_0 \in \mathcal{C}(\mathbf{z}_0) \tag{5.21}$$

Hence b_0 is a candidate and Theorem 5.1 is proved. □

Next a useful corollary states sufficient conditions for completeness. The conditions of this corollary is easier to check then the condition of Theorem 5.1.

Corollary 5.2. *Given a diagnostic model* $\mathbb{M}$ *and a diagnostic system* Δ*,* Δ *is complete with respect to* $\mathbb{M}$ *if all tests* δ_i *are designed such that*

$$\Phi_i \subseteq \operatorname{ass} M_i \tag{5.22}$$

and

$$\mathcal{O}^{\mathbb{M}}_{M_i} \subseteq \mathcal{O}^{\Delta}_{\delta_i} \tag{5.23}$$

Before we prove Corollary 5.2 a Lemma is presented and proved.

Lemma 5.3. *Given a diagnostic model* $\mathbb{M}$*, a set of system behavioral modes* $\Phi \subseteq \mathcal{B}$*, and a model* $M \subseteq^* \mathbb{M}$ *it follows that*

$$\Phi \subseteq \operatorname{ass} M \leftrightarrow (M \subseteq M_\Phi) \tag{5.24}$$

First we recall the definitions

$$M_\Phi := \{e | \Phi \subseteq \operatorname{ass} e\} \tag{5.25}$$

and

$$\operatorname{ass} M = \bigcap_{e \in M} \operatorname{ass} e \tag{5.26}$$

Proof. We begin to prove the right direction of (5.24). Assume that

$$\Phi \subseteq \operatorname{ass} M \tag{5.27}$$

Expression (5.27) is rewritten using (5.26) as

$$\Phi \subseteq \bigcap_{e \in M} \operatorname{ass} e \tag{5.28}$$

From (5.28) it follows that

$$\forall e \in M : \Phi \subseteq \operatorname{ass} e \tag{5.29}$$

Expression (5.25) and (5.29) imply that

$$M \subseteq M_\Phi \tag{5.30}$$

The conclusion of (5.27)-(5.30) is that

$$\Phi \subseteq \operatorname{ass} M \rightarrow M \subseteq M_\Phi \tag{5.31}$$

Now, we prove the left implication of (5.24). Assume that

$$M \subseteq M_\Phi \tag{5.32}$$

From the definition in (5.26) it follows that

$$M \subseteq M_\Phi \rightarrow \operatorname{ass} M_\Phi \subseteq \operatorname{ass} M \tag{5.33}$$

Expressions (5.32) and (5.33) imply that

$$\operatorname{ass} M_\Phi \subseteq \operatorname{ass} M \tag{5.34}$$

From the definition (5.25) it follows that

$$\Phi \subseteq \operatorname{ass} M_\Phi \tag{5.35}$$

and using (5.34) implies that

$$\Phi \subseteq \operatorname{ass} M \tag{5.36}$$

The conclusion from the calculations done in (5.32)-(5.36) is that

$$M \subseteq M_\Phi \rightarrow \Phi \subseteq \operatorname{ass} M \tag{5.37}$$

From (5.31) and (5.37), expression (5.24) follows. □

Now, we prove Corollary 5.2.

Proof. For $b \in \Phi_i$ it follows that $M_{\Phi_i} \subseteq M_b$. This implies that $\mathcal{O}^{\mathbb{M}}_{M_b} \subseteq \mathcal{O}^{\mathbb{M}}_{M_{\Phi_i}}$. Since this is true for all $b \in \Phi_i$ it follows that

$$\bigcup_{b \in \Phi_i} \mathcal{O}^{\mathbb{M}}_{M_b} \subseteq \mathcal{O}^{\mathbb{M}}_{M_{\Phi_i}} \tag{5.38}$$

From (5.22) and Lemma 5.3 it follows that

$$\mathcal{O}^{\mathbb{M}}_{M_{\Phi_i}} \subseteq \mathcal{O}^{\mathbb{M}}_{M_i} \tag{5.39}$$

Theorem 5.1 together with (5.38), (5.39), and (5.23) completes the proof. □

The most sensitive test to check validity of M_i appears when equality holds for expressions (5.22) and (5.23). However a common situation is that $\mathcal{O}^{\mathbb{M}}_{M_i}$ only approximate the true behavior of the system. Therefore, to be sure that $\mathcal{O}^{\Delta}_{\delta_i}$ is a superset to the observations of the true behavior, a larger set has to be chosen. In expression (5.22) the weakest and only reasonable choice is $\Phi_i = \operatorname{ass} M_i$.

Example 5.1 Consider the diagnostic model and the diagnostic system in Example 2.6. The two tests δ_1 and δ_2 satisfy condition (5.22) and (5.23). Hence according to Corollary 5.2 the diagnostic system is complete.

From now on we assume that all tests are designed such that the considered diagnostic systems are complete.

5.2 Examples of Complete and Sound Diagnostic System

A diagnostic system is complete if each test is designed such that (5.2) is fulfilled. That is, completeness is independent on which set of models $M \subseteq^* \mathbb{M}$ that are checked. In this section we present conditions for a complete diagnostic system to be sound. It is not just a question of how tests are designed, it is also important which set of models that are checked.

5.2.1 Model Definitions

First we need some definitions to characterize different types of models.

Definition 5.1 (Feasible Model). *Given a diagnostic model* $\mathbb{M}$*, a model* $M \subseteq \mathbb{M}$ *is a* ***feasible model*** *if* $\text{ass}\, M \neq \varnothing$*.*

From now on all models that are mentioned are feasible models.

Definition 5.2 (Rejectable Model at $\mathbf{z}_0$). *Given a diagnostic model* $\mathbb{M}$*, a (feasible) model* $M \subseteq^* \mathbb{M}$ *is a* ***rejectable model at*** $\mathbf{z}_0$ *if*

$$\forall \mathbf{x} \neg M(\mathbf{x}, \mathbf{z}_0) \tag{5.40}$$

Definition 5.3 (Minimal Rejectable Model at $\mathbf{z}_0$). *Given a diagnostic model* $\mathbb{M}$*, a rejectable model* $M \subseteq^* \mathbb{M}$ *at* $\mathbf{z}_0$ *is a* ***minimal rejectable model at*** $\mathbf{z}_0$ *if no proper subset to* M *is a rejectable model at* $\mathbf{z}_0$*.*

Definition 5.4 (Minimal Rejectable Model). *Given a diagnostic model* $\mathbb{M}$*, a model* M *is a* ***minimal rejectable model*** *if there is a* $\mathbf{z}$ *such that* M *is a minimal rejectable model at* $\mathbf{z}$*.*

The set of all minimal rejectable models in $\mathbb{M}$ at $\mathbf{z}_0$ is denoted $\gamma_m(\mathbf{z}_0)$ and the set of all minimal rejectable models in $\mathbb{M}$ is denoted γ_m. Next we exemplify the different types of models in a diagnostic model.

Example 5.2 Continuation of Example 2.6. In Table 2.3 the diagnostic model is given. To get the a nontrivial example the diagnostic model is extended with two new equations such that

component	assumption	equation
Sensor 1	$\phi(s_1 = NF)$	$e_1: \quad z_1 = x_1$
Comp	$\mathcal{B}$	$e_2: \quad x_1 = x_2^2$
Sensor 2	$\phi(s_2 = NF)$	$e_3: \quad z_2 = x_2$
Sensor 3	$\phi(s_3 = NF)$	$e_4: \quad z_3 = x_2$
	$\phi(s_3 = SG)$	$e_5: \quad z_3 = 0$

(5.41)

and

component	behavioral modes
Sensor 1	$s_1 \in \{NF, UF\}$
Sensor 2	$s_2 \in \{NF, UF\}$
Sensor 3	$s_3 \in \{NF, SG\}$

(5.42)

The model $\{e_4, e_5\}$ in (5.41) is not a feasible model because

$$\text{ass}\{e_4, e_5\} = \varnothing \tag{5.43}$$

The model $\{e_5\}$ is a feasible model. Let $\mathbf{z}_0 = (z_1, z_2, z_3)$ be such that $z_1 < 0$ and $z_2 = z_3 \neq 0$. The (feasible) rejectable models at $\mathbf{z}_0$ are

$$\begin{array}{l} \{e_5\}, \{e_1, e_2\}, \{e_1, e_5\}, \{e_2, e_5\}, \{e_3, e_5\}, \\ \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_5\}, \{e_1, e_3, e_5\}, \\ \{e_2, e_3, e_5\}, \{e_1, e_2, e_3, e_4\}, \{e_1, e_2, e_3, e_5\} \end{array} \tag{5.44}$$

An example of a model that is not a rejectable model at $\mathbf{z}_0$ is $\{e_3, e_4\}$. The minimal rejectable models at $\mathbf{z}_0$ are

$$\gamma_m(\mathbf{z}_0) = \{\{e_1, e_2\}, \{e_5\}\} \tag{5.45}$$

The rejectable model $\{e_1, e_5\}$ at $\mathbf{z}_0$ is not a minimal rejectable model at $\mathbf{z}_0$ because the proper subset $\{e_5\}$ is a rejectable model at $\mathbf{z}_0$.

The minimal rejectable models are

$$\gamma_m = \{\{e_5\}, \{e_3, e_4\}, \{e_1, e_2\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\} \tag{5.46}$$

Note that both $\{e_1, e_2\}$ and $\{e_1, e_2, e_3\}$ are minimal rejectable models even though $\{e_1, e_2\} \subset \{e_1, e_2, e_3\}$. To explain this, let $\mathbf{z}_1 = (z_1, z_2, z_3)$ be such that $z_1 \geq 0$, $z_1 \neq z_2^2$, and $z_2 = z_3 \neq 0$. The minimal rejectable models at $\mathbf{z}_1$ are

$$\gamma_m(\mathbf{z}_1) = \{\{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_5\}\} \tag{5.47}$$

The model $\{e_1, e_2, e_3\}$ is a minimal rejectable model at $\mathbf{z}_1$ and one of its subset $\{e_1, e_2\}$ is a minimal rejectable model at $\mathbf{z}_0$. According to Definition 5.4 it holds that both $\{e_1, e_2\}$ and $\{e_1, e_2, e_3\}$ are minimal rejectable models. An example of a model that is not a minimal rejectable model is $\{e_1, e_2, e_3, e_4\}$. This model is for example rejectable at $\mathbf{z}_0$ as seen in (5.44) but if it is rejectable at any $\mathbf{z}$ there is always a proper subset that is rejectable at $\mathbf{z}$ too. It is for example $\{e_1, e_2\}$ that is a minimal rejectable model at $\mathbf{z}_0$.

In the next sections we will show two sets of tests that can be used to design a sound diagnostic system.

5.2.2 Check Behavioral Models

To compare the diagnoses and the candidates both the diagnostic model and the diagnostic system have to be defined. The diagnostic system is designed using the diagnostic model. The tests are assumed to be of the type defined next.

Definition 5.5 (Strong Diagnostic Test for M). *A test δ_i is a **strong diagnostic test for** a model M if*

$$\Phi_i := \operatorname{ass} M \tag{5.48}$$

and

$$\mathcal{O}_{\delta_i} := \mathcal{O}_M \tag{5.49}$$

Example 5.3 The two diagnostic tests δ_1 and δ_2 in Example 2.6 are examples of strong diagnostic tests for M_1 and M_2 respectively.

Note that strong tests are in practice often not possible to use because there are uncertainties and noise that influence the physical system. Therefore the diagnostic model is often only a good approximation of the physical system. Next a condition is given to design a sound and complete diagnostic system.

Theorem 5.4. *Given a diagnostic model $\mathbb{M}$, a diagnostic system Δ is complete and sound if there is a strong test δ_i for each system behavioral model M_b.*

Before we prove Theorem 5.4 an example show how the design is done.

Example 5.4 The behavioral models in (5.41) are

b	M_b
$\langle NF, NF, NF\rangle$	$\{e_1, e_2, e_3, e_4\}$
$\langle UF, NF, NF\rangle$	$\{e_2, e_3, e_4\}$
$\langle NF, UF, NF\rangle$	$\{e_1, e_2, e_4\}$
$\langle NF, NF, SC\rangle$	$\{e_1, e_2, e_3, e_5\}$
$\langle UF, UF, NF\rangle$	$\{e_2, e_4\}$
$\langle UF, NF, SC\rangle$	$\{e_2, e_3, e_5\}$
$\langle NF, UF, SC\rangle$	$\{e_1, e_2, e_5\}$
$\langle UF, UF, SC\rangle$	$\{e_2, e_5\}$

(5.50)

According to Theorem 5.4, a complete and sound diagnostic system for (5.41) is

Δ	M_i	$H_i^0 : sys \in \Phi_i = \text{ass}\, M_i$	$\mathcal{O}_{\delta_i}^{\Delta} = \mathcal{O}_{M_i}$
δ_1	$\{e_1, e_2, e_3, e_4\}$	$\phi(s_1 = NF \wedge s_2 = NF \wedge s_3 = NF)$	$\{\mathbf{z} \vert z_1 = z_2^2, z_2 = z_3\}$
δ_2	$\{e_2, e_3, e_4\}$	$\phi(s_2 = NF \wedge s_3 = NF)$	$\{\mathbf{z} \vert z_2 = z_3\}$
δ_3	$\{e_1, e_2, e_4\}$	$\phi(s_1 = NF \wedge s_3 = NF)$	$\{\mathbf{z} \vert z_1 = z_3^2\}$
δ_4	$\{e_1, e_2, e_3, e_5\}$	$\phi(s_1 = NF \wedge s_2 = NF \wedge s_3 = SC)$	$\{\mathbf{z} \vert z_1 = z_2^2, z_3 = 0\}$
δ_5	$\{e_2, e_4\}$	$\phi(s_2 = NF)$	$\mathbb{R}^3$
δ_6	$\{e_2, e_3, e_5\}$	$\phi(s_2 = NF \wedge s_3 = SC)$	$\{\mathbf{z} \vert z_3 = 0\}$
δ_7	$\{e_1, e_2, e_5\}$	$\phi(s_1 = NF \wedge s_3 = SC)$	$\{\mathbf{z} \vert z_1 \geq 0, z_3 = 0\}$
δ_8	$\{e_2, e_5\}$	$\phi(s_3 = SG)$	$\{\mathbf{z} \vert z_3 = 0\}$

(5.51)

Note that in general is $\text{ass}\, M_b \neq \{b\}$. For example in δ_2 it holds that

$$\text{ass}\, M_{\langle UF,NF,NF\rangle} = \{\langle NF, NF, NF\rangle, \langle UF, NF, NF\rangle\} \neq \{\langle UF, NF, NF\rangle\} \tag{5.52}$$

However it holds that $b \in \text{ass}\, M_b$. The null hypothesis of test δ_5 is not be rejectable since $\mathcal{O}_{\delta_5}^{\Delta} = \mathbb{R}^3$. Therefore it is possible to omit δ_5.

Proof. All tests are strong tests and a strong test fulfill both (5.22) and (5.23). Hence the conditions in Corollary 5.2 are fulfilled and it follows that Δ is complete with respect to $\mathbb{M}$. Now it remains to prove that Δ is sound with respect to $\mathbb{M}$, i.e.

$$\forall \mathbf{z}(\mathcal{C}(\mathbf{z}) \subseteq \mathcal{D}(\mathbf{z})) \tag{5.53}$$

Chose an arbitrary $\mathbf{z} = \mathbf{z}_0$ and an arbitrary system behavioral mode called b_1 such that

$$b_1 \in \mathcal{C}(\mathbf{z}_0) \tag{5.54}$$

Enumerate the behavioral modes b_i and design a strong test for each M_{b_i} such that

$$\mathcal{O}^{\Delta}_{\delta_i} := \mathcal{O}^{\mathbb{M}}_{M_{b_i}} \tag{5.55}$$

and

$$\Phi_i := \text{ass}\, M_{b_i} \tag{5.56}$$

Assume that

$$b_1 \notin \mathcal{D}(\mathbf{z}_0) \tag{5.57}$$

or equivalently

$$\mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M_{b_1}} \tag{5.58}$$

From (5.55) and (5.58) it follows that

$$\mathbf{z}_0 \notin \mathcal{O}^{\Delta}_{\delta_1} \tag{5.59}$$

Further using (5.56) it is easy to realize that

$$b_1 \in \text{ass}\, M_{b_1} = \Phi_1 \tag{5.60}$$

From (5.3), (5.59), and (5.60) it follows that

$$b_1 \notin \mathcal{C}(\mathbf{z}_0) \subseteq \Phi_1^C \tag{5.61}$$

Expression (5.54) and (5.61) is a contradiction. It follows that $b_1 \in \mathcal{D}(\mathbf{z})$. Since $\mathbf{z}$ and $b_1 \in \mathcal{C}(\mathbf{z})$ were arbitrarily chosen the theorem follows. □

If exactly the diagnostic system (5.51) was implemented we would for example check if $z_3 = 0$ in 4 out of the 8 tests. This is computationally not an efficient way to diagnose the system. Larger models are more difficult to evaluate in general. Behavioral models have the disadvantage of being the largest feasible models. These two reasons rise the question if there exist alternative sound and complete diagnostic systems using fewer and simpler tests?

5.2.3 Check Minimal Rejectable Models

A sound and complete diagnostic system can be constructed designing tests for a the minimal rejectable models of a diagnostic model. As the name indicates they are the smallest models that can be used. Hence test quantities are often more easily obtained.

Theorem 5.5. *Given a diagnostic model* $\mathbb{M}$*, a diagnostic system* Δ *is complete and sound if there is a strong test* δ_i *for each minimal rejectable model* $M_i \subseteq^* \mathbb{M}$*.*

Example 5.5 The minimal rejectable models in (5.41) are

$$\gamma_m = \{\{e_1, e_2\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_3, e_4\}, \{e_5\}\} \tag{5.62}$$

A complete and sound diagnostic system for (5.41) is

Δ	$H_i^0 : \Phi_i = \text{ass}\, M_i$	M_i	$\mathcal{O}_{\delta_i}^{\Delta} = \mathcal{O}_{M_i}$
δ_1	$\phi(s_1 = \mathsf{NF})$	$\{e_1, e_2\}$	$\{\mathbf{z} \vert z_1 \geq 0\}$
δ_2	$\phi(s_1 = \mathsf{NF} \wedge s_2 = \mathsf{NF})$	$\{e_1, e_2, e_3\}$	$\{\mathbf{z} \vert z_1 = z_2^2\}$
δ_3	$\phi(s_1 = \mathsf{NF} \wedge s_3 = \mathsf{NF})$	$\{e_1, e_2, e_4\}$	$\{\mathbf{z} \vert z_1 = z_3^2\}$
δ_4	$\phi(s_2 = \mathsf{NF} \wedge s_3 = \mathsf{NF})$	$\{e_3, e_4\}$	$\{\mathbf{z} \vert z_2 = z_3\}$
δ_5	$\phi(s_3 = \mathsf{SG})$	$\{e_5\}$	$\{\mathbf{z} \vert z_3 = 0\}$

(5.63)

A comparison between (5.51) and (5.63) reveals that the number of tests in (5.63) is smaller, the tests in (5.63) contains less number of equations, and the sets $\mathcal{O}_{\delta_i}^{\Delta}$ in (5.63) is described easier.

Proof. From (5.48) and (5.49), expression (5.22) and (5.23) follows respectively. Hence according to Corollary 5.2, Δ is complete with respect to $\mathbb{M}$. It remains to prove that Δ is sound with respect to $\mathbb{M}$. Assume the contrary, i.e. there is a $\mathbf{z}_0$ such that b is a candidate but b is not a diagnosis. From the definition of diagnosis it follows that

$$\mathbf{z}_0 \notin \mathcal{O}_{M_b}^{\mathbb{M}} \tag{5.64}$$

From the definition of minimal rejectable model at $\mathbf{z}_0$ there exist a minimal rejectable model M_1 at $\mathbf{z}_0$ such that

$$M_1 \subseteq M_b \wedge \mathbf{z}_0 \notin \mathcal{O}_{M_1}^{\mathbb{M}} \tag{5.65}$$

Note that the existence follows from the fact that if no proper subset to M_b has property (5.65) then M_b is a minimal rejectable model at $\mathbf{z}_0$. Since all minimal rejectable models at $\mathbf{z}_0$ are minimal rejectable models it follows from the conditions stated in Theorem 5.5 that there is a strong test δ_1 for M_1. From (5.49) and (5.65) it follows that

$$\mathbf{z}_0 \notin \mathcal{O}_{\delta_1}^{\Delta} \tag{5.66}$$

Hence H_1^0 is rejectable and from (5.3) it follows that

$$\mathcal{C}(\mathbf{z}_0) \subseteq \Phi_1^C \tag{5.67}$$

Now, (5.48) implies that

$$\Phi_1 = \text{ass}\, M_1 \tag{5.68}$$

and (5.65) implies that

$$b \in \text{ass}\, M_b \subseteq \text{ass}\, M_1 \tag{5.69}$$

Hence (5.67), (5.68), and (5.69) gives that

$$b \notin \mathcal{C}(\mathbf{z}_0) \tag{5.70}$$

i.e. b is not a candidate. This contradict the assumption and it follows that Δ is sound. □

5.3 Sound and Complete Diagnostic Systems

Finally we would like to give a if and only if condition of the set of models that can be used to derive a sound diagnostic system. It turns out that the condition also implies that the diagnostic system is complete. Hence it is also a if and only if condition of the set of models that can be used to design a sound and complete diagnostic system. To be able to state the condition, some definitions are needed.

Definition 5.6 (Detection Model for M_b). *Given a diagnostic model $\mathbb{M}$ and a behavioral model $M_b \subseteq^* \mathbb{M}$, a minimal rejectable model $M \subseteq M_b$ is a* ***detection model for*** M_b *if no proper superset to M, that is a subset of M_b, is a minimal rejectable model.*

Let the set of all detection models for M_b be denoted Σ_b. Let $\sigma_b \subseteq \Sigma_b$ be a minimal set such that

$$\bigcap_{M \in \Sigma_b} \mathcal{O}_M^{\mathbb{M}} = \bigcap_{M \in \sigma_b} \mathcal{O}_M^{\mathbb{M}} \tag{5.71}$$

Theorem 5.6 (Sound Diagnostic System). *Given a diagnostic model $\mathbb{M}$, let γ be a set of models $M \subseteq^* \mathbb{M}$. Let a strong test for each model in γ define a diagnostic system Δ. Then the diagnostic system Δ is sound with respect to the diagnostic model $\mathbb{M}$ if and only if γ fulfills*

$$\forall b \in \mathcal{B}(\exists \sigma_b \forall M' \in \sigma_b \exists M \in \gamma : M' \subseteq M \subseteq M_b) \tag{5.72}$$

An especially interesting type of γ that satisfies (5.72) is those γ:s that only contain minimal models such that (5.72) is fulfilled.

Corollary 5.7. *Given a diagnostic model $\mathbb{M}$, let γ be a set of models $M \subseteq^* \mathbb{M}$. Let a strong test for each model in γ define a diagnostic system Δ. The minimal models M, that can be used to define a diagnostic system Δ that is sound with respect to the diagnostic model $\mathbb{M}$, are minimal rejectable models.*

The proof of Corollary 5.7 follows directly from Theorem 5.6, the definitions of σ_b, Σ_b, and detection model. Now we prove Theorem 5.6. Two examples follow after the proof.

Proof. We start to show that (5.72) is a sufficient condition for soundness. Hence we will prove that

$$\forall \mathbf{z}(\mathcal{C}(\mathbf{z}) \subseteq \mathcal{D}(\mathbf{z})) \tag{5.73}$$

holds if (5.72) is satisfied. Let $\mathbf{z}_0$ be an arbitrary $\mathbf{z}$ and let b be an arbitrary

$$b_0 \in \mathcal{C}(\mathbf{z}_0) \tag{5.74}$$

We will show that b_0 is a diagnosis. From the definition of diagnostic statement it follows that

$$b_0 \in \mathcal{C}(\mathbf{z}_0) = \bigcap_{\mathbf{z}_0 \notin \mathcal{O}_{\delta_i}^{\Delta}} \Phi_i^C \cap \mathcal{B} \tag{5.75}$$

From (5.75) it follows that

$$\mathbf{z}_0 \notin \mathcal{O}^{\Delta}_{\delta_i} \rightarrow b_0 \notin \Phi_i \tag{5.76}$$

The implied expression in (5.76) can be rewritten using the fact that there is a strong test for each model $M_i \in \gamma$ according to

$$b_0 \notin \Phi_i \leftrightarrow b_0 \notin \text{ass}\, M_i \leftrightarrow M_i \nsubseteq M_{b_0} \tag{5.77}$$

The last equivalence in (5.77) follows from Lemma 5.3 by setting $\Phi = \{b_0\}$ in (5.24). Using (5.76) and (5.77) it follows that

$$\mathbf{z}_0 \notin \mathcal{O}^{\Delta}_{\delta_i} \rightarrow M_i \nsubseteq M_{b_0} \tag{5.78}$$

Since δ_i is a strong test it holds that

$$M_i \subseteq M_{b_0} \rightarrow \mathbf{z}_0 \in \mathcal{O}^{\mathbb{M}}_{M_i} \tag{5.79}$$

From (5.72) it follows that there is a σ_{b_0} according to (5.71) such that

$$\forall M \in \sigma_{b_0} \exists M_i \in \gamma : M \subseteq M_i \subseteq M_{b_0} \tag{5.80}$$

holds. Expression (5.79), and (5.80) imply that

$$\forall M \in \sigma_{b_0} : \mathbf{z}_0 \in \mathcal{O}^{\mathbb{M}}_{M} \tag{5.81}$$

From Definition 5.6 and (5.71) it follows that

$$\mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M_{b_0}} \rightarrow \exists M \in \sigma_{b_0} : \mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M} \tag{5.82}$$

or equivalently

$$\forall M \in \sigma_{b_0} : \mathbf{z}_0 \in \mathcal{O}^{\mathbb{M}}_{M} \rightarrow \mathbf{z}_0 \in \mathcal{O}^{\mathbb{M}}_{M_{b_0}} \tag{5.83}$$

Finally from expression (5.81), and (5.83) it follows that

$$\mathbf{z}_0 \in \mathcal{O}^{\mathbb{M}}_{M_{b_0}} \tag{5.84}$$

Hence $b_0 \in \mathcal{D}(\mathbf{z}_0)$. Since $\mathbf{z}_0$ and b_0 were arbitrarily chosen the sufficient direction follows. Now, the necessary direction remains to prove. We will prove the equivalent statement, if (5.72) does not hold then $\exists \mathbf{z} : \mathcal{C}(\mathbf{z}) \nsubseteq \mathcal{D}(\mathbf{z})$. Assume that (5.72) does not hold, i.e.

$$\exists b \in \mathcal{B} \forall \sigma_b \exists M' \in \sigma_b \forall M_i \in \gamma : M' \nsubseteq M_i \vee M_i \nsubseteq M_b \tag{5.85}$$

Let a behavioral mode b and an arbitrary σ_b and an M' that fulfill (5.85) be b_0, σ_{b_0}, and M'_0 respectively. Since $M'_0 \in \sigma_{b_0}$ it holds using (5.71) that there exists a $\mathbf{z}_0$ such that

$$\mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M'_0} \wedge \forall M' \in \sigma_{b_0} : M' \neq M'_0 \rightarrow \mathbf{z}_0 \in \mathcal{O}^{\mathbb{M}}_{M'} \tag{5.86}$$

From $M_0' \in \sigma_{b_0}$ it follows that $M_0' \subseteq M_{b_0}$. Then it is true that

$$\mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M_0'} \rightarrow \mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M_{b_0}} \tag{5.87}$$

Using (5.86) and (5.87) it is clear that

$$b_0 \notin \mathcal{D}(\mathbf{z}_0) \tag{5.88}$$

Now we will investigate if b_0 is a candidate. Since δ_i are strong test for the models $M_i \in \gamma$ the candidates are calculated as

$$\mathcal{C}(\mathbf{z}_0) = \bigcap_{\substack{M_i \subseteq M_{b_0} \\ \mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M_i}}} \Phi_i^C \cap \bigcap_{\substack{M_i \nsubseteq M_{b_0} \\ \mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M_i}}} \Phi_i^C \cap \mathcal{B} \tag{5.89}$$

If we start to look at the first intersection in (5.89), where $M_i \subseteq M_{b_0}$. We know that $M_0' \subseteq M_{b_0}$ because $M_0' \in \sigma_{b_0}$. Furthermore M_0' satisfy (5.85), that is

$$M_i \in \gamma : M_i \subseteq M_{b_0} \rightarrow M_0' \nsubseteq M_i \tag{5.90}$$

From Definition 5.6 and (5.86) it follows that

$$M_0' \subseteq M_i \leftrightarrow \mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M_i} \tag{5.91}$$

Now from (5.90) and (5.91) it follows that

$$\forall M_i \in \gamma : M_i \subseteq M_{b_0} \rightarrow \mathbf{z}_0 \in \mathcal{O}^{\mathbb{M}}_{M_i} \tag{5.92}$$

Hence the candidates can be rewritten as

$$\mathcal{C}(\mathbf{z}_0) = \bigcap_{\substack{M_i \nsubseteq M_{b_0} \\ \mathbf{z}_0 \notin \mathcal{O}^{\mathbb{M}}_{M_i}}} \Phi_i^C \cap \mathcal{B} \tag{5.93}$$

Now, looking at those $M_i \in \gamma$ where $M_i \nsubseteq M_{b_0}$. From Lemma 5.3 and since only strong tests are used

$$M_i \nsubseteq M_{b_0} \leftrightarrow b \notin \operatorname{ass} M_i = \Phi_i \tag{5.94}$$

From (5.93) and (5.94) it follows that

$$b_0 \in \bigcap_{M_i \nsubseteq M_{b_0}} \Phi_i^C \cap \mathcal{B} \subseteq \mathcal{C}(\mathbf{z}_0) \tag{5.95}$$

From (5.88) we have that $b_0 \notin \mathcal{D}(\mathbf{z}_0)$ and from (5.95) that $b_0 \in \mathcal{C}(\mathbf{z}_0)$. Hence $\mathcal{C}(\mathbf{z}_0) \nsubseteq \mathcal{D}(\mathbf{z}_0)$ and the other direction follows. □

Example 5.6 Consider the diagnostic model in (5.41). The detection models for each system behavioral mode are shown in Table 5.1. In this example $\sigma_b = \Sigma_b$ for all behavioral modes except for $b = b_1$ where

$$\sigma_{b_1} = \{\{e_1, e_2, e_3\}, \{e_3, e_4\}\} \tag{5.97}$$

Table 5.1 *The set of detection models for each system behavioral mode in the diagnostic model defined in (5.41).*

b	M_b	Σ_b
b_1	$\{e_1, e_2, e_3, e_4\}$	$\{\{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_3, e_4\}\}$
b_2	$\{e_2, e_3, e_4\}$	$\{\{e_3, e_4\}\}$
b_3	$\{e_1, e_2, e_4\}$	$\{\{e_1, e_2, e_4\}\}$
b_4	$\{e_1, e_2, e_3, e_5\}$	$\{\{e_1, e_2, e_3\}, \{e_5\}\}$
b_5	$\{e_2, e_4\}$	$\emptyset$
b_6	$\{e_2, e_3, e_5\}$	$\{\{e_5\}\}$
b_7	$\{e_1, e_2, e_5\}$	$\{\{e_1, e_2\}, \{e_5\}\}$
b_8	$\{e_2, e_5\}$	$\{\{e_5\}\}$

(5.96)

or

$$\sigma_{b_1} = \{\{e_1, e_2, e_4\}, \{e_3, e_4\}\} \tag{5.98}$$

According to Theorem 5.6 a complete and sound diagnostic system is obtained if and only if (5.72) holds. Two particular sets γ that we studied earlier were $\gamma_B = \{M_b | b \in \mathcal{B}\}$ and γ_m. Starting with $\gamma = \gamma_B$, (5.72) trivially holds. Since all detection models for a behavioral mode are minimal rejectable models it follows that the diagnostic system is complete and sound when $\gamma = \gamma_m$.

Theorem 5.6 can be used to find the minimal number of tests that have to be used to design a sound and complete diagnostic system. The minimal number of tests for the diagnostic model described in (5.41) is 5. This can be realized from the following discussion. Row b_3 in Table 5.1 and condition (5.72) imply that there must be a set $M \in \gamma$ such that $\{e_1, e_2, e_4\} \subseteq M \subseteq \{e_1, e_2, e_4\}$. Hence a sound and complete diagnostic system must include a strong test of $\{e_1, e_2, e_4\}$. Row b_8 implies that either $\{e_5\}$ or $\{e_2, e_5\}$ must be included in γ. Since $\{e_2, e_5\} \notin \Sigma_b$ for any $b \in \mathcal{B}$, $\{b_5\}$ can be chosen. With $\gamma = \{\{e_5\}, \{e_1, e_2, e_4\}\}$ condition (5.72) of b_3, b_5, b_6, and b_8 are fulfilled. Continuing in this way the minimum number of 5 tests must be used to fulfill all conditions on γ. The diagnostic system using all minimal rejectable model shown in (5.63) is an example of a sound and complete diagnostic system with only 5 tests.

Note also that Theorem 5.6 can be used to design a sound and complete diagnostic system for a subset of system behavioral modes. Exchange $\mathcal{B}$ in (5.72) with a set $A \subseteq \mathcal{B}$. If the modified condition (5.72) is fulfilled then the diagnostic system will be sound and complete with respect to the behavioral modes in A but only complete with respect of the behavioral modes not included in A. This can be expressed as

$$\forall \mathbf{z} : A \cap C(\mathbf{z}) = A \cap D(\mathbf{z}) \tag{5.99}$$

and

$$\forall \mathbf{z} : (\mathcal{B} \backslash A) \cap D(\mathbf{z}) \subseteq (\mathcal{B} \backslash A) \cap C(\mathbf{z}) \tag{5.100}$$

Next an example will show the special case when A is the set of all single faults and no-fault.

Example 5.7 Assume that $A = \{b_1, b_2, b_3, b_4\}$. The minimal number of tests that has to be used is 3. One example is $\gamma = \{\{e_3, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_3, e_5\}\}$. Note that $\{e_1, e_2, e_3, e_5\}$ is not a minimal rejectable model. If only minimal rejectable models are used, 4 tests are needed. If behavioral models are used, 4 tests are also needed. Assume that $sys = b_2$, that is the first sensor is broken. It has been observed that $z_1 < 0$. The only minimal rejectable model for this $\mathbf{z}$ is assumed to be $\{e_1, e_2\}$. This implies that

$$\mathcal{D}(\mathbf{z}) = \{b_2, b_5, b_6, b_8\} \tag{5.101}$$

and

$$\mathcal{C}(\mathbf{z}) = \{b_2, b_5, b_6, b_7, b_8\} \tag{5.102}$$

Expression (5.99) is clearly fulfilled in this example since

$$A \cap C(\mathbf{z}) = \{b_2\} = A \cap D(\mathbf{z}) \tag{5.103}$$

Furthermore

$$(\mathcal{B} \backslash A) \cap D(\mathbf{z}) = \{b_5, b_6, b_8\} \tag{5.104}$$

and

$$(\mathcal{B} \backslash A) \cap C(\mathbf{z}) = \{b_5, b_6, b_7, b_8\} \tag{5.105}$$

imply that (5.100) is fulfilled. Note that b_7 is a candidate in (5.105) but not a diagnosis in (5.104). Hence this diagnostic system is not sound with respect to all behavioral modes.

6

Isolability Analysis of Diagnostic Models

In Chapter 4 we defined structural and analytical isolability of a diagnostic system. It was shown that structural isolability is a necessary condition for analytical isolability of a diagnostic system. Then a structural method was suggested to calculate the structural isolability of a diagnostic system. In this chapter we extend the two definitions analytical and structural isolability of diagnostic systems to be valid also for diagnostic models. The *analytical isolability of a diagnostic model* $\mathbb{M}$ is the best possible analytical isolability of a diagnostic system designed using the diagnostic model $\mathbb{M}$. Then we present a structural method that calculates the *structural isolability of a diagnostic model* $\mathbb{M}$. In the same way as in Chapter 4, it will be shown that the structural isolability of a diagnostic model is a necessary condition for the analytical isolability of a diagnostic model.

The structural isolability of a diagnostic model is calculated in a two step approach. Later we will see that with this two step approach, results from Chapter 5 extend easily the method to calculate the isolability of a diagnostic system presented in Chapter 4, to be applicable also to diagnostic models.

In the first step a set of models γ is calculated using the diagnostic model such that Theorem 5.6 is fulfilled. Since Theorem 5.6 is fulfilled it follows that γ represents a sound and complete diagnostic system with respect to the analyzed diagnostic model. In this chapter it will be shown that a sound and complete diagnostic system with respect to the diagnostic model $\mathbb{M}$ has the same analytical isolability as the diagnostic model $\mathbb{M}$. Since the analytical isolability of the diagnostic model is equal to the analytical isolability of a diagnostic system represented by a γ satisfying Theorem 5.6, the structural isolability of the diagnostic system represented by γ is a necessary condition also for the analytical isolability of the diagnostic model.

In the second step the structural isolability of the diagnostic system represented by γ is calculated. These calculations have been defined in Chapter 4. According to Definition 4.1 the set of Φ_i defines the structural isolability of a diagnostic system. This set can be obtained directly from the set γ and the diagnostic model $\mathbb{M}$, i.e. there is no need to calculate the diagnostic system represented by γ to calculate the structural isolability of the diagnostic system.

In Section 6.1 structural and analytical isolability of a diagnostic model is defined. In Section 6.2 we give a sufficient condition of γ:s that implies that the structural isolability is a necessary condition for the analytical isolability. Then we give two important examples of γ:s, i.e. all behavioral models and all minimal rejectable models. These two sets γ imply different structural isolability. In Section 6.3 a condition on γ is given that implies the best structural isolability. In Section 6.4 different options are discussed to calculate a γ that fulfills the condition in Section 6.3 without using analytical properties. Finally in Section 6.5 the analytical isolability of a diagnostic model is calculated. The computationally complexity of calculating the analytical isolability is significantly decreased using the structural isolability computed in previous sections.

6.1 Structural and Analytical Isolability of a Diagnostic Model

We start to define analytical isolability of a diagnostic model.

Definition 6.1 ($\mathcal{I}^{\mathbb{M}}$, Analytical Isolability of a Diagnostic Model). *If $\mathbb{M}$ is a diagnostic model, then the **analytical isolability of the diagnostic model** $\mathbb{M}$ is a binary relation $\mathcal{I}^{\mathbb{M}}$ on $\mathcal{B} \times \mathcal{B}$ defined as*

$$\mathcal{I}^{\mathbb{M}} := \{(b_1, b_2) | \exists \mathbf{z} : \big(b_1 \in \mathcal{D}(\mathbf{z}) \wedge b_2 \notin \mathcal{D}(\mathbf{z})\big)\} \tag{6.1}$$

If $(b_1, b_2) \in \mathcal{I}^{\mathbb{M}}$ we say that b_1 is *isolable* from b_2 with the diagnostic model $\mathbb{M}$. Definition 6.1 defines which behavioral modes that are analytically isolable from each other given a model $\mathbb{M}$. The calculation of the analytical isolability requires that analytical properties of $\mathbb{M}$ are known. The analytical isolability of a model $\mathbb{M}$, $\mathcal{I}^{\mathbb{M}}$, is interesting, because it is equal to the best analytical isolability of any diagnostic system designed using $\mathbb{M}$. Often it is difficult to exactly calculate $\mathcal{I}^{\mathbb{M}}$ given the diagnostic model $\mathbb{M}$, but the structural properties of $\mathbb{M}$ are easily obtained. Later it will be clear that necessary conditions for the analytical isolability can be derived using only structural properties. Next we use the definition of structural isolability of a diagnostic system to define the structural isolability of a diagnostic model.

Definition 6.2 ($\mathcal{I}_s^{\mathbb{M}}(\gamma)$, Structural Isolability of a Diagnostic Model). *If $\mathbb{M}$ is a diagnostic model, γ is a set of models contained in $\mathbb{M}$, and Δ is a diagnostic system such that there is a strong test δ_i for each $M \in \gamma$, then the **structural***

isolability of the diagnostic model $\mathbb{M}$ *is a binary relation* $\mathcal{I}_s^{\mathbb{M}}(\gamma)$ *on* $\mathcal{B} \times \mathcal{B}$ *defined as*

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) := \mathcal{I}_s^{\Delta} \tag{6.2}$$

The idea behind Definition 6.2 is that if b_1 is structurally isolable from b_2, then there exists a model M that can reject b_2 but not b_1. Note that in Definition 6.2, γ is an arbitrary set of models contained in $\mathbb{M}$. However, it is only for some sets γ that the structural isolability is a necessary condition for the analytical isolability. Sufficient conditions for such sets of models γ will be presented later.

If γ in (6.2) can be calculated structurally, then only the structural properties of $\mathbb{M}$ are needed to calculate $\mathcal{I}_s^{\mathbb{M}}(\gamma)$. According to Definition 6.2 it is straightforward to calculate $\mathcal{I}_s^{\mathbb{M}}(\gamma)$ as soon as γ is known. The structural isolability $\mathcal{I}_s^{\mathbb{M}}(\gamma)$ is calculated by putting $H_i^0 : \Phi_i = \text{ass}\, M_i$ for each $M_i \in \gamma$ and then applying Algorithm 4.1. Combining it all together gives that

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) = \bigcup_{M \in \gamma} \{(b_1, b_2) | b_1 \notin \text{ass}\, M \land b_2 \in \text{ass}\, M\} \tag{6.3}$$

In Theorem 4.5 it was stated that larger sets Φ_i give less optimistic structural isolability, i.e. a stronger restriction of the analytical isolability. Since $\Phi_i = \text{ass}\, M_i$, the size of Φ_i is determined from $\text{ass}\, M_i$ that is derived from $\text{ass}\, e$ where $e \in M_i$. During the design of the diagnostic model, behavioral mode assumptions $\text{ass}\, e$ are used to imply e. After the design of a diagnostic model it can be worthwhile to analyze if system behavioral modes can be added to the some $\text{ass}\, e$ defined during the design. This is desirable because if any $\text{ass}\, e$ can be increased, the sets Φ_i can become larger and the structural isolability of the model becomes less optimistic.

6.2 Structural Isolability Necessary Condition for Analytical Isolability

In Definition 6.2, γ plays an important role to define the structural isolability of a diagnostic model. The next lemma and theorem gives sufficient conditions on γ such that the structural isolability of a diagnostic model is a necessary condition for the analytical isolability of the analyzed diagnostic model.

Lemma 6.1. *If* Δ *is a sound and complete diagnostic system with respect to a diagnostic model* $\mathbb{M}$ *it follows that*

$$\mathcal{I}^{\mathbb{M}} = \mathcal{I}^{\Delta} \tag{6.4}$$

Proof. Since Δ is a sound and complete diagnostic system with respect to a diagnostic model $\mathbb{M}$ it means that

$$\forall \mathbf{z} : (\mathcal{D}(\mathbf{z}) = \mathcal{C}(\mathbf{z})) \tag{6.5}$$

Definition 6.1 implies that

$$\mathcal{I}^{\mathbb{M}} = \{(b_1, b_2) | \exists \mathbf{z} : \big(b_1 \in \mathcal{D}(\mathbf{z}) \land b_2 \notin \mathcal{D}(\mathbf{z})\big)\} \tag{6.6}$$

From (6.5) and (6.6) it follows that

$$\mathcal{I}^{\mathbb{M}} = \{(b_1, b_2) | \exists \mathbf{z} : \big(b_1 \in \mathcal{C}(\mathbf{z}) \wedge b_2 \notin \mathcal{C}(\mathbf{z})\big)\} \tag{6.7}$$

Expression (4.2) in Definition 4.2 and (6.7) imply (6.4). □

Theorem 6.2. *Given a diagnostic model* $\mathbb{M}$ *and a* γ *that fulfills Theorem 5.6, it follows that*

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma) \tag{6.8}$$

Proof. Since $\mathbb{M}$ and γ fulfill Theorem 5.6 a diagnostic system Δ is defined such that it is sound and complete with respect to $\mathbb{M}$. Lemma 6.1 implies that

$$\mathcal{I}^{\mathbb{M}} = \mathcal{I}^{\Delta} \tag{6.9}$$

From Theorem 4.1 it holds that

$$\mathcal{I}^{\Delta} \subseteq \mathcal{I}_s^{\Delta} \tag{6.10}$$

Using (6.2), (6.9) and (6.10) it follows that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma) \tag{6.11}$$

□

A sufficient condition of the choice of γ is that $\mathbb{M}$ and γ fulfill Theorem 5.6. In Chapter 5 two important examples of γ that fulfill Theorem 5.6 were shown, namely γ_B from Theorem 5.4 and γ_m from Theorem 5.5. In the next two sections, these two choices of γ will be discussed and exemplified.

6.2.1 Structural Isolability Using System Behavioral Models

The first choice to be considered is the set of all system behavioral models of a diagnostic model, i.e. $\gamma = \gamma_B$ from Section 5.2.2. The set γ_B is of special interest because it is easy to calculate for any diagnostic model and fulfills Theorem 5.6. Later we will see that the disadvantage of using γ_B is that the structural isolability is more optimistic than for other choices of γ. Next we state that $\gamma = \gamma_B$ implies that the structural isolability is a necessary condition for the analytical isolability.

Corollary 6.3. *Given a diagnostic model* $\mathbb{M}$ *and*

$$\gamma_B = \{M_b | b \in \mathcal{B}\} \tag{6.12}$$

it holds that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma_B) \tag{6.13}$$

Proof. The result follows directly from Theorem 5.4 and Theorem 6.2. □

In this special case when $\gamma = \gamma_B$, the expression (6.3) can be simplified. A particular simple expression appears for the set complement of $\mathcal{I}_s^{\mathbb{M}}(\gamma_B)$, i.e.

$$\overline{\mathcal{I}_s^{\mathbb{M}}}(\gamma_B) = \{(b_i, b_j) | b_i \in \text{ass}\, M_{b_j}\} \tag{6.14}$$

Note that this expression can directly be applied to any diagnostic model. The structural isolability $\overline{\mathcal{I}_s^{\mathbb{M}}}(\gamma_B)$ can be calculated as follows.

Algorithm 6.1.
Input: $\mathcal{B}$ *and* ass e *for all* e *in* $\mathbb{M}$.

a) *Set* $\mathcal{I}_s^{\mathbb{M}}(\gamma_B) := \varnothing$.

b) *For each system behavioral-mode* $b_j \in \mathcal{B}$, *set*

$$\overline{\mathcal{I}_s^{\mathbb{M}}}(\gamma_B) := \overline{\mathcal{I}_s^{\mathbb{M}}}(\gamma_B) \cup \{(b_i, b_j) | b_i \in \text{ass}\, M_{b_j}\} \tag{6.15}$$

Output: $\overline{\mathcal{I}_s^{\mathbb{M}}}(\gamma_B)$

An example now shows how $\overline{\mathcal{I}_s^{\mathbb{M}}}(\gamma_B)$ is calculated using Algorithm 6.1.

Example 6.1 Continuation of Example 5.4. Recall that the diagnostic model is defined as

component	assumption	equation
Sensor 1	$\phi(s_1 = NF)$	$e_1: \; z_1 = x_1$
Comp	$\mathcal{B}$	$e_2: \; x_1 = x_2^2$
Sensor 2	$\phi(s_2 = NF)$	$e_3: \; z_2 = x_2$
Sensor 3	$\phi(s_3 = NF)$	$e_4: \; z_3 = x_2$
	$\phi(s_3 = SG)$	$e_5: \; z_3 = 0$

(6.16)

and

component	behavioral modes
Sensor 1	$s_1 \in \{NF, UF\}$
Sensor 2	$s_2 \in \{NF, UF\}$
Sensor 3	$s_3 \in \{NF, SG\}$

(6.17)

Let the system behavioral modes be enumerated as

$$\begin{array}{l} b_1 = \langle NF, NF, NF \rangle \\ \hline b_2 = \langle UF, NF, NF \rangle \\ b_3 = \langle NF, UF, NF \rangle \\ b_4 = \langle NF, NF, SG \rangle \\ \hline b_5 = \langle UF, UF, NF \rangle \\ b_6 = \langle UF, NF, SG \rangle \\ b_7 = \langle NF, UF, SG \rangle \\ \hline b_8 = \langle UF, UF, SG \rangle \end{array}$$

When Algorithm 6.1 is applied to this diagnostic model the calculated structural isolability is

present mode	necessary interpreted mode							
	b_1	b_2	b_3	b_5	b_4	b_6	b_7	b_8
b_1	X	X	X	X				
b_2		X		X				
b_3			X	X				
b_5				X				
b_4					X	X	X	X
b_6						X		X
b_7							X	X
b_8								X

(6.18)

According to Algorithm 6.1 the first iteration calculates the column in (6.18) corresponding to behavioral mode b_1 in $I_s^{\mathbb{M}}(\gamma_B)$. In this column there will be an "X" in each row corresponding to behavioral modes in $\text{ass}\, M_{b_1}$. The set $\text{ass}\, M_{b_1}$ is calculated in two steps. First M_{b_1} is found to be

$$M_{b_1} = \{e_1, e_2, e_3, e_4\} \tag{6.19}$$

Then $\text{ass}\, M_{b_1}$ is calculated as

$$\text{ass}\, M_{b_1} = \text{ass}\,(\{e_1, e_2, e_3, e_4\}) = \phi(s_1 = \mathsf{NF} \wedge s_2 = \mathsf{NF} \wedge s_3 = \mathsf{NF}) = \{b_1\} \tag{6.20}$$

The first column in (6.18) is the result of $\text{ass}\, M_{b_1} = \{b_1\}$. The next behavioral mode b_2 calculates the next column in $I_s^{\mathbb{M}}(\gamma_B)$. The behavioral mode b_2 has the model

$$M_{b_2} = \{e_2, e_3, e_4\}$$

From this model the assumption is obtained as

$$\text{ass}\, M_{b_2} = \text{ass}\,(\{e_2, e_3, e_4\}) = \phi(s_2 = \mathsf{NF} \wedge s_3 = \mathsf{NF}) = \{b_1, b_2\} \tag{6.21}$$

These calculations are repeated for all system behavioral modes $\mathcal{B}$. The isolability matrix $I_s^{\mathbb{M}}(\gamma_B)$ can be interpreted in two different ways. First assume that a complete but not necessarily sound diagnostic system is designed using γ_B. Furthermore, assume for example that b_1 is the present system behavioral mode, i.e. the diagnosed system is working in behavioral mode b_1 and the observation $\mathbf{z}_0 \in \mathcal{O}_{b_1}$ is observed. Then (6.18) implies that all modes where the corresponding column is denoted with an "X" in the row corresponding to b_1 are candidates. For this example it means that $\{b_1, b_2, b_3, b_5\} \subseteq \mathcal{C}(\mathbf{z}_0)$. Since γ_B fulfills Theorem 4.1 it follows that (6.18) can also be interpreted as $\{b_1, b_2, b_3, b_5\} \subseteq \mathcal{D}(\mathbf{z}_0)$. This means that given any complete diagnostic system for this diagnostic model, $\{b_1, b_2, b_3, b_5\}$ must be candidates when b_1 is the present mode. Note that, if b_1 is the present behavioral mode, then (6.18) does not imply that b_4, b_6, b_7, and b_8 are not diagnosis or candidates depending on the interpretation. The Hasse diagram of the corresponding partial order $\mathcal{P}_s^{\mathbb{M}}(\gamma_B)$ is shown in Figure 6.1.

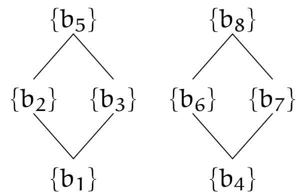

Figure 6.1 *The Hasse diagram of the partial order* $\mathcal{P}_s^{\mathbb{M}}(\gamma_B)$.

Among the component behavioral modes, it is common that the unknown fault component behavioral mode UF is included. Since the behavior of this component behavioral mode is unknown, it is for example impossible to isolate NF from UF. This information does not need to be added separately to the structural isolability analysis, because it is contained in the structural isolability as the next example shows.

Example 6.2 The structural isolability analysis can be done separately on each component. Consider Sensor 1 in the diagnostic model (6.16). Calculating the structural isolability using Algorithm 6.1 involves the following computations

$$\begin{aligned} \text{ass}\, M_{\mathsf{NF}} &= \text{ass}\{e_1\} = \{\mathsf{NF}\} \\ \text{ass}\, M_{\mathsf{UF}} &= \text{ass}\, \varnothing = \{\mathsf{NF}, \mathsf{UF}\} \end{aligned} \tag{6.22}$$

that gives

present	necessary interpreted mode	
mode	NF	UF
NF	X	X
UF		X

(6.23)

It is clear that NF is not isolable from UF. Finally we analyze the connection between the structural isolability analysis of a component and the structural isolability analysis of the entire diagnostic model. Since we know the isolability of the model of Sensor 1, it implies that for any pair of $s_2 \in \{\mathsf{NF}, \mathsf{UF}\}$ and $s_3 \in \{\mathsf{NF}, \mathsf{SG}\}$ it follows that

present	necessary interpreted mode	
mode	$\langle \mathsf{NF}, s_2, s_3 \rangle$	$\langle \mathsf{UF}, s_2, s_3 \rangle$
$\langle \mathsf{NF}, s_2, s_3 \rangle$	X	X
$\langle \mathsf{UF}, s_2, s_3 \rangle$		X

(6.24)

The implication of (6.24) in the structural isolability matrix (6.18) is marked with four boxes of size 2×2 in

present mode	necessary interpreted mode							
	b_1	b_2	b_3	b_5	b_4	b_6	b_7	b_8
b_1	X	X	X	X				
b_2		X		X				
b_3			X	X				
b_5				X				
b_4					X	X	X	X
b_6						X		X
b_7							X	X
b_8								X

(6.25)

In (6.24) we can for example put $s_2 = \text{NF}$ and $s_3 = \text{NF}$ and get a relation between the behavioral mode $b_1 = \langle \text{NF}, \text{NF}, \text{NF} \rangle$ and $b_2 = \langle \text{UF}, \text{NF}, \text{NF} \rangle$. In (6.25), the upper-left box corresponds two b_1 and b_2. This box and (6.24) is equal. If $\mathbb{M}$ is a diagnostic model where c is one of the components, then $\mathbb{M}(c)$ denotes the *diagnostic model for component* c. If the diagnostic model contains the components $c_1, \ldots, c_n$ then

$$\mathcal{I}_s^{\mathbb{M}}(\gamma_B) = \mathcal{I}_s^{\mathbb{M}(c_1)}(\gamma_B) \times \ldots \times \mathcal{I}_s^{\mathbb{M}(c_n)}(\gamma_B) \tag{6.26}$$

That is the structural isolability of the diagnostic model $\mathbb{M}$ can be obtained combining the structural isolability of each component model.

6.2.2 Structural Isolability Using Minimal Rejectable Models

Another choice of γ is to use the set of all minimal rejectable models γ_m described in Section 5.2.3. This γ is of special interest because it gives, as we will see later, often a less optimistic structural isolability than for example γ_B. The disadvantage is that γ_m cannot be used in a purely structural analysis, because analytical properties of the diagnostic model are used to calculate γ_m.

Corollary 6.4. *For a diagnostic model* $\mathbb{M}$ *it holds that*

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma_m) \tag{6.27}$$

Proof. The result follows directly from Theorem 5.5 and Theorem 6.2. □

In the next example the structural isolability of the diagnostic model (6.16) is calculated when $\gamma = \gamma_m$.

Example 6.3 The set of all minimal rejectable models in the diagnostic model (6.16) was presented in Example 5.5. From (5.63) in Example 5.5 a sound and complete

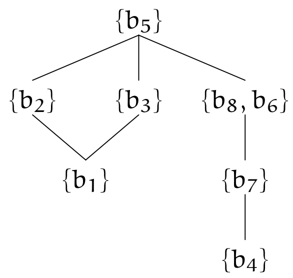

Figure 6.2 *The Hasse diagram of the partial order* $\mathcal{P}_s^{\mathbb{M}}(\gamma_m)$.

diagnostic system is

Δ	M_i	$\Phi_i = \text{ass}\, M_i$	$\mathcal{O}_{\delta_i}^{\Delta} = \mathcal{O}_{M_i}$
δ_1	$\{e_1, e_2\}$	$\phi(s_1 = NF)$	$\{\mathbf{z} \vert z_1 \geq 0\}$
δ_2	$\{e_1, e_2, e_3\}$	$\phi(s_1 = NF \wedge s_2 = NF)$	$\{\mathbf{z} \vert z_1 = z_2^2\}$
δ_3	$\{e_1, e_2, e_4\}$	$\phi(s_1 = NF \wedge s_3 = NF)$	$\{\mathbf{z} \vert z_1 = z_3^2\}$
δ_4	$\{e_3, e_4\}$	$\phi(s_2 = NF \wedge s_3 = NF)$	$\{\mathbf{z} \vert z_2 = z_3\}$
δ_5	$\{e_5\}$	$\phi(s_3 = SG)$	$\{\mathbf{z} \vert z_3 = 0\}$

(6.28)

where a test is designed for each minimal rejectable model. Calculating (6.3) when $\gamma = \gamma_m$ gives that the structural isolability of the diagnostic model $\mathcal{I}_s^{\mathbb{M}}(\gamma_m)$ is

present	necessary interpreted mode						
mode	$\{b_1\}$	$\{b_2\}$	$\{b_3\}$	$\{b_4\}$	$\{b_7\}$	$\{b_8, b_6\}$	$\{b_5\}$
$\{b_1\}$	X	X	X				X
$\{b_2\}$		X					X
$\{b_3\}$			X				X
$\{b_4\}$				X	X	X	X
$\{b_7\}$					X	X	X
$\{b_8, b_6\}$						X	X
$\{b_5\}$							X

(6.29)

The corresponding partial order is shown in Figure 6.2.

6.2.3 Comparison of Structural Isolability for Different γ:s

Comparing (6.18) and (6.29) reveals some differences. For example, there is a difference in the column corresponding to $b_5 = \langle UF, UF, NF \rangle$. The analytical expressions of $M_{b_5} = \{e_2, e_4\}$ are $x_1 = x_2^2$ and $z_3 = x_2$. Any z_3 will satisfy these two constraints, i.e. the model M_{b_5} is not rejectable. Then according to the definition of diagnosis it follows that b_5 is always a diagnosis.

In Example 6.3 we know all minimal rejectable models γ_m. All models that are rejectable are equal to, or a superset to, a minimal rejectable model. In (6.28) the minimal rejectable models are shown in the "M_i" column. If these models are compared with M_{b_5}, it is clear that M_{b_5} is not equal to, or a superset to, any minimal rejectable model. This implies that M_{b_5} is not rejectable at any $\mathbf{z}$. Since M_{b_5} is not rejectable it will also with this argument follow that b_5 always is a diagnosis. The structural isolability given in Example 6.3 also states that b_5 always is a diagnosis. This can be seen in (6.29) where there is an "X" in every row of the column corresponding to b_5. This means that b_5 is a diagnosis independent of the present mode. Hence b_5 is always a diagnosis.

This conclusion cannot be obtained from the structural isolability (6.18) in Example 6.1. In the column corresponding at b_5 in (6.18), we have for example not ruled out that b_4 is isolable from b_5. If b_4 is isolable from b_5 then a necessary condition is that there exists a test δ_i, such that $b_4 \notin \Phi_i$ and $b_5 \in \Phi_i$. In Example 6.1, the tests are defined with $\gamma = \gamma_B$. A test that fulfill both requirements is δ_5 with $\Phi_5 = \text{ass}\, M_{b_5} = \{b_1, b_2, b_3, b_5\}$ where b_4 is not included. When the structural isolability is calculated it is assumed that the null hypothesis of each test can be rejected. Since M_{b_5} is not rejectable at any $\mathbf{z}$ the structural analysis in Example 6.1 gets more optimistic than the structural analysis in Example 6.3.

If (6.18) and (6.29) are inspected it is clear that

$$\mathcal{I}_s^{\mathbb{M}}(\gamma_m) \subset \mathcal{I}_s^{\mathbb{M}}(\gamma_B) \tag{6.30}$$

Note that the diagnostic system derived using γ_B and γ_m are both sound and complete with respect to a diagnostic model, i.e. they have exactly the same analytical isolability. In spite of the fact that they have the same analytical isolability, we have seen an example that shows that they can have different structural isolability.

6.3 Structural Isolability as a Function of γ

The structural isolability is obviously dependent on which γ that is chosen among those satisfying (5.72). From Theorem 6.2 we know that for all γ that fulfill (5.72) it holds that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma) \tag{6.31}$$

Since we know that all γ that fulfill (5.72) have the desired property (6.31) and that the structural isolability is dependent on which of these sets γ that is used, the resulting question is which of these sets γ that gives the least optimistic structural isolability. Before the next Theorem is stated that characterize these sets γ of models, an assumption is presented.

Assumption 6.1. *Let $\mathbb{M}$ be a diagnostic model such that for all behavioral modes $b \in \mathcal{B}$ it holds that*

$$\forall \sigma_b (\bigcup_{M \in \sigma_b} M = \bigcup_{M \in \Sigma_b} M) \tag{6.32}$$

where Σ_b and σ_b is defined in Section 5.3.

Assumption 6.1 can be interpreted, as for each behavioral mode there exists a unique minimal set of equations, $\bigcup_{M\in\Sigma_b} M$, that describes the same set of possible observations $\mathbf{z}$ as M_b.

Assumption 6.1 is often fulfilled, for example the for the diagnostic model described in Example 6.1. For this diagnostic model, the sets Σ_b and σ_b is calculated in Example 5.6. Using the calculated Σ_b:s and σ_b:s in Example 5.6, condition (6.32) is verified and it follows that Assumption 6.1 holds. To simplify the notation let

$$M_b^* := \bigcup_{M\in\Sigma_b} M \tag{6.33}$$

Theorem 6.5 (Structural Isolability of a Diagnostic Model). *Let $\mathbb{M}$ be a diagnostic model that fulfills Assumption 6.1 and γ and γ' two sets of models satisfying Theorem 5.6. If γ fulfills*

$$\forall b \in \mathcal{B}\big(\bigcup_{\substack{M \in \gamma \\ M \subseteq M_b}} M = M_b^*\big) \tag{6.34}$$

then it follows that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma) \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma') \tag{6.35}$$

Proof. Since γ and γ' fulfill (5.72) it follows from Theorem 6.2 that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma) \tag{6.36}$$

and

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma') \tag{6.37}$$

respectively. Now, it remains to prove that

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma') \tag{6.38}$$

Take an arbitrary

$$(b_1, b_2) \in \mathcal{I}_s^{\mathbb{M}}(\gamma) \tag{6.39}$$

From (6.3) and (6.39) it follows that

$$\exists M \in \gamma : b_1 \notin \operatorname{ass} M \wedge b_2 \in \operatorname{ass} M \tag{6.40}$$

Using Lemma 5.3, expression (6.40) can be rewritten as

$$\exists M \in \gamma : M \not\subseteq M_{b_1} \wedge M \subseteq M_{b_2} \tag{6.41}$$

Let $\bar{M}$ be a model that satisfy (6.41). Then from (6.34) it follows that

$$\bar{M} \not\subseteq M_{b_1} \wedge \bar{M} \subseteq M_{b_2}^* \tag{6.42}$$

This implies that

$$M_{b_2}^* \backslash M_{b_1} \neq \emptyset \tag{6.43}$$

Since it holds that γ' fulfills (5.72), there is a σ_{b_2} such that

$$\forall M' \in \sigma_{b_2} \exists M \in \gamma' : M' \subseteq M \subseteq M_{b_2} \tag{6.44}$$

From Assumption 6.1 and (6.43) it follows that there is a $\hat{M} \in \sigma_{b_2}$ such that

$$\hat{M} \not\subseteq M_{b_1} \land \hat{M} \subseteq M_{b_2} \tag{6.45}$$

Expression (6.44) and (6.45) imply that there exists an $\tilde{M} \in \gamma'$ such that

$$\tilde{M} \not\subseteq M_{b_1} \land \tilde{M} \subseteq M_{b_2} \tag{6.46}$$

Using Lemma 5.3, (6.46) is rewritten as

$$\exists \tilde{M} \in \gamma' : b_1 \notin \text{ass}\,\tilde{M} \land b_2 \in \text{ass}\,\tilde{M} \tag{6.47}$$

This means according to (6.3) that

$$(b_1, b_2) \in \mathcal{I}_s^{\mathbb{M}}(\gamma') \tag{6.48}$$

Hence from (6.39) and (6.48), it follows that

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma') \tag{6.49}$$

□

Example 6.4 Once again we will compare the structural isolability obtained if γ_B is used as in Example 6.1 and if γ_m is used as in Example 6.3. However the differences will this time be discussed in light of Theorem 6.5. First note that condition (6.32) follows from Example 5.6. It is clear that γ_m fulfills the condition (6.34) and from Theorem 6.5 it follows that the structural isolability is the least optimistic structural isolability that can be obtained. Contrary γ_B does not fulfill (6.34). This can be seen in

b_i	M_{b_i}		$M^*_{b_i}$
b_1	$\{e_1, e_2, e_3, e_4\}$	$=$	$\{e_1, e_2, e_3, e_4\}$
b_2	$\{e_2, e_3, e_4\}$	$\neq$	$\{e_3, e_4\}$
b_3	$\{e_1, e_2, e_4\}$	$=$	$\{e_1, e_2, e_4\}$
b_4	$\{e_1, e_2, e_3, e_5\}$	$=$	$\{e_1, e_2, e_3, e_5\}$
b_5	$\{e_2, e_4\}$	$\neq$	$\emptyset$
b_6	$\{e_2, e_3, e_5\}$	$\neq$	$\{e_5\}$
b_7	$\{e_1, e_2, e_5\}$	$=$	$\{e_1, e_2, e_5\}$
b_8	$\{e_2, e_5\}$	$\neq$	$\{e_5\}$

(6.50)

The condition (6.34) is fulfilled for the behavioral modes b_1, b_3, b_4, and b_7. Interpreting Theorem 6.5 and (6.50) gives that the structural isolability in (6.18) is the best in the columns corresponding to b_1, b_3, b_4, and b_7. Since Theorem 6.5 only gives sufficient conditions for optimal isolability it does not state that the columns corresponding to b_2, b_5, b_6, and b_8 are not optimal. To find out if these columns are equal we have to compare these columns in (6.18) with the corresponding columns in (6.29) which is the best structural isolability. It is clear that apart from b_1, b_3, b_4, and b_7 the column corresponding to b_2 in (6.18) is identical to the column corresponding to b_2 in (6.29).

6.4 Structural Isolability Using Structural Properties

In Theorem 6.5 it was shown how to chose γ among those γ:s that fulfill Theorem 5.6 to derive the best structural isolability. However Σ_b in condition (6.34) is often not known and to compute Σ_b detailed analytical analyses are needed to be done. Therefore it is not easy to directly use (6.34) as a design criteria for γ. In this section we will discuss which γ that implies the best structural isolability given a certain level of knowledge about the diagnostic model.

6.4.1 The Best Structural Isolability Given a Certain Level of Knowledge

If Σ_b in condition (6.34) is not known, γ can not be chosen to fulfill (6.34) in Theorem 6.5 to obtain the best structural isolability $\mathcal{I}_s^{\mathbb{M}}(\gamma)$. An alternative is to use a structural method, as will be explained later, to get a γ that corresponds to a good structural isolability. As before we study those γ:s that fulfill (5.72) in Theorem 5.6, which implies according to Theorem 6.2 that the structural isolability is a necessary condition for the analytical isolability. To design a structural method that finds a γ that can be used to get good structural isolability, it is important to understand how the structural isolability varies depending on γ. This section will give details of how the structural isolability varies depending on γ. Then in the following sections two different structural methods will be suggested.

Dependence between γ and Structural Isolability

Before a Theorem is presented, that characterize the dependence between γ and $\mathcal{I}_s^{\mathbb{M}}(\gamma)$, some useful notation is introduced. Let C_b be an arbitrary model such that $C_b \subseteq M_b$. Sometimes a set of models γ will define particular C_b:s. Then the notation $C_b(\gamma)$ will be used and is defined as

$$C_b(\gamma) := \bigcup_{\substack{M \in \gamma \\ M \subseteq M_b}} M \tag{6.51}$$

The model $C_b(\gamma)$ is the set of equations that a diagnostic system defined by γ uses to check inconsistencies for behavioral mode b. A model C_b, not defined with a γ, can be thought of as the set of equations that a diagnostic system uses to check inconsistencies for behavioral mode b. The model M_b^* is the smallest set of equations that exactly defines the set of possible observations for behavior mode b.

It will especially for theoretical purposes be convenient to define a *primitive structural isolability* as

$$\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_1}, \dots, C_{b_n}\rangle) := \{(b_i, b_j) \in \mathcal{B} \times \mathcal{B} | b_i \notin \text{ass}\, C_{b_j}\} \quad (6.52)$$

where C_{b_i} is an arbitrary model such that $C_{b_i} \subseteq M_{b_i}$. The difference between $\mathcal{I}_s^{\mathbb{M}}(\{C_{b_1}, \dots, C_{b_n}\})$ and $\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_1}, \dots, C_{b_n}\rangle)$ is in the latter case that, if the model C_{b_i} is not valid then only b_i is rejected as the next example will show.

Example 6.5 Continuation of Example 6.4. Consider the following three choices of C_b:s corresponding to each column in

b	$C^1_{b_i} = M^*_{b_i}$	$C^2_{b_i}$	$C^3_{b_i} = M_{b_i}$
b_1	$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_3, e_4\}$
b_2	$\{e_2, e_3, e_4\}$	$\{e_2, e_3, e_4\}$	$\{e_2, e_3, e_4\}$
b_3	$\{e_1, e_2, e_4\}$	$\{e_1, e_2, e_4\}$	$\{e_1, e_2, e_4\}$
b_4	$\{e_1, e_2, e_3, e_5\}$	$\{e_1, e_2, e_3, e_5\}$	$\{e_1, e_2, e_3, e_5\}$
b_5	$\varnothing$	$\{e_2, e_4\}$	$\{e_2, e_4\}$
b_6	$\{e_5\}$	$\{e_5\}$	$\{e_2, e_3, e_5\}$
b_7	$\{e_1, e_2, e_5\}$	$\{e_1, e_2, e_5\}$	$\{e_1, e_2, e_5\}$
b_8	$\{e_5\}$	$\{e_2, e_5\}$	$\{e_2, e_5\}$

(6.53)

Consider $C^1_{b_6} = \{e_5\}$ where ass $C^1_{b_6} = \{b_4, b_6, b_7, b_8\}$. The model $C^1_{b_6}$ contributes according to (6.52) to the primitive structural isolability as

$$\{(b_i, b_6) | b_i \notin \text{ass}\, C^1_{b_6}\} = \{(b_i, b_6) | b_i \in \{b_1, b_2, b_3, b_5\}\} \quad (6.54)$$

and in the structural isolability defined in (6.3) as

$$\begin{aligned} &\{(b_i, b_j) | b_i \notin \text{ass}\, C^1_{b_6} \wedge b_j \in \text{ass}\, C^1_{b_6}\} = \\ &\{(b_i, b_j) | b_i \in \{b_1, b_2, b_3, b_5\} \wedge b_j \in \{b_4, b_6, b_7, b_8\}\} \end{aligned} \quad (6.55)$$

These two sets can be represented in the isolability matrix as

present	necessary interpreted mode							
mode	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
b_1				0		**0**	0	0
b_2				0		**0**	0	0
b_3				0		**0**	0	0
b_4								
b_5				0		**0**	0	0
b_6								
b_7								
b_8								

(6.56)

where the zeros corresponds to the set defined by (6.55) and the bold zeros denote the set in (6.54). The primitive structural isolability only draw conclusions about the column corresponding to b_6, i.e. if $C^1_{b_6} = \{e_5\}$ is not valid then b_6 is not a diagnosis. This means that the null hypothesis for $C^1_{b_6}$ in this case is $sys = b_6$. For the standard structural isolability the weakest behavioral mode assumption is used as a null hypothesis, i.e. $sys \in \text{ass}\, C^1_{b_6} = \{b_4, b_6, b_7, b_8\}$.

The next theorem presents basic properties needed to understand the relation between different sets γ and their corresponding structural isolability.

Theorem 6.6. *Let* $\mathbb{M}$ *be a diagnostic model, let* $b_1, \ldots, b_n$ *be the system behavioral modes, and let* $\langle C_{b_1}, \ldots, C_{b_n} \rangle$ *be a tuple of one model for each behavioral mode such that* $C_{b_i} \subseteq M_{b_i}$. *Then the following holds:*

a) It holds that $\mathcal{I}^{\mathbb{M}}_{sp}(\langle C_{b_1}, \ldots, C_{b_n} \rangle) \subseteq \mathcal{I}^{\mathbb{M}}_{s}(\{C_{b_1}, \ldots, C_{b_n}\})$.

b) If there exists a γ *such that* $(C_{b_1}, \ldots, C_{b_n}) = (C_{b_1}(\gamma), \ldots, C_{b_n}(\gamma))$*, then* $\mathcal{I}^{\mathbb{M}}_{s}(\gamma) = \mathcal{I}^{\mathbb{M}}_{s}(\{C_{b_1}, \ldots, C_{b_n}\}) = \mathcal{I}^{\mathbb{M}}_{sp}(\langle C_{b_1}, \ldots, C_{b_n} \rangle)$.

c) If γ *is a set of models that satisfy the condition (6.34) in Theorem 6.5, if and only if* $\forall b_i \in \mathcal{B} : C_{b_i}(\gamma) = M^*_{b_i}$ *holds.*

d) If $C_{b_i} \subseteq C'_{b_i}$ *holds for all* $b_i \in \mathcal{B}$*, then* $\mathcal{I}^{\mathbb{M}}_{sp}(\langle C_{b_1}, \ldots, C_{b_n} \rangle) \subseteq$ $\subseteq \mathcal{I}^{\mathbb{M}}_{sp}(\langle C'_{b_1}, \ldots, C'_{b_n} \rangle)$.

e) If $\mathbb{M}$ *fulfills Assumption 6.1 and* $M^*_{b_i} \subseteq C_{b_i}$ *holds for all* $b_i \in \mathcal{B}$*, then there exists a* γ *such that* $\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}^{\mathbb{M}}_{s}(\gamma) \subseteq \mathcal{I}^{\mathbb{M}}_{sp}(\langle C_{b_1}, \ldots, C_{b_n} \rangle)$.

Before Theorem 6.6 will be proven a discussion and an example will illustrate how the results are used. The goal is, as said before, to derive the least optimistic structural isolability that is a necessary condition for the analytical isolability. Items (b), (c), and (d) in Theorem 6.6 imply that the primitive structural isolability is a necessary condition for the analytical isolability if

$$\forall b \in \mathcal{B} : M^*_b \subseteq C_b \subseteq M_b \tag{6.57}$$

holds. It can be realized in the following way. Using item (c), Theorem 6.2 and Theorem 6.5 it follows that $\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}^{\mathbb{M}}_{s}(\{M^*_b\})$. From items (b) and (c) it follows that $\mathcal{I}^{\mathbb{M}}_{s}(\{M^*_b\}) = \mathcal{I}^{\mathbb{M}}_{sp}(\langle M^*_b \rangle)$. Item (d) implies that $\mathcal{I}^{\mathbb{M}}_{sp}(\langle M^*_b \rangle) \subseteq \mathcal{I}^{\mathbb{M}}_{sp}(\langle C_b \rangle)$ if $M^*_b \subseteq C_b$. Finally, from the definition of C_b it follows that $C_b \subseteq M_b$. It is clear that all three choices of $C^j_{b_i}$ in (6.53) fulfill (6.57) , i.e. $M^*_{b_i} \subseteq C^j_{b_i} \subseteq M_{b_i}$ for all system behavioral modes b_i and for all three cases $j \in \{1, 2, 3\}$. This implies as said before that $\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}^{\mathbb{M}}_{sp}(\langle C^j_{b_i} \rangle)$.

To get the least optimistic structural isolability, item (d) in Theorem 6.6 implies that the smallest C_b that fulfills (6.57), i.e. M^*_b, corresponds to the best structural isolability. Note that M^*_b in general is unknown. However it is easier to conclude that a set C_b fulfills $M^*_b \subseteq C_b$ than to calculate M^*_b, as will be shown later in Section 6.4.3. Consider a case where M^*_b is unknown, but it is assumed that there is a set of C_b for each system behavioral mode b that fulfill (6.57). If for each behavioral mode b, the smallest C_b among the considered models are chosen, the best structural isolability among the considered models will be obtained according to item (d) in Theorem 6.6. Consider the three choices of $C^j_{b_i}$ in (6.53). The primitive structural isolability of case 1 is

present mode	necessary interpreted mode							
	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
b_1	X	X	X		X			
b_2		X			X			
b_3			X		X			
b_4				X	X	X	X	X
b_5					X			
b_6					X	X		X
b_7					X	X	X	X
b_8					X	X		X

(6.58)

in case 2

present mode	necessary interpreted mode							
	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
b_1	X	X	X		X			
b_2		X			X			
b_3			X		X			
b_4				X		X	X	X
b_5					X			
b_6						X		X
b_7						X	X	X
b_8						X		X

(6.59)

and in case 3

present mode	necessary interpreted mode							
	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
b_1	X	X	X		X			
b_2		X			X			
b_3			X		X			
b_4				X		X	X	X
b_5					X			
b_6						X		X
b_7							X	X
b_8								X

(6.60)

Item (d) is illustrated noting that $C^1_{b_i} \subseteq C^2_{b_i} \subseteq C^3_{b_i}$ and comparing (6.58), (6.59), and (6.60). For example, the difference between the case 2 and 3 is that $C^2_{b_6} = \{e_5\}$ but $C^3_{b_6} = \{e_2, e_3, e_5\}$. From the calculation of $\mathcal{I}^{\mathbb{M}}_{sp}(\langle C_{b_i} \rangle)$ this difference implies that the column corresponding to b_6 can differ. The difference of (6.59) and (6.60) is that (b_7, b_6) and (b_8, b_6) is not in the isolability matrix (6.60). Remember that the isolability matrix is defined as the complement set to the isolability relation. Then it follows from (6.59) and (6.60) that $\mathcal{I}^{\mathbb{M}}_{sp}(\langle C^2_{b_i} \rangle) \subseteq \mathcal{I}^{\mathbb{M}}_{sp}(\langle C^3_{b_i} \rangle)$. If all three isolability matrices are compared, it is easy to see that smaller sets C_b:s give less optimistic structural isolability, i.e. larger numbers of "X":s in the isolability matrices.

When the best possible tuple of C_b:s is chosen, item (a) in Theorem 6.6 suggests that the primitive structural isolability $\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_i}\rangle)$ gives a less optimistic structural isolability than the structural isolability $\mathcal{I}_{s}^{\mathbb{M}}(\{C_{b_i}\})$. Item (e) implies that even if $\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_i}\rangle)$ gives a less optimistic structural isolability, it is still a necessary condition for the analytical isolability.

Finally item (c) and (e) imply that the least optimistic structural isolability using a γ defined in Theorem 6.5 is the same as the least optimistic structural isolability calculated as $\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_i}\rangle)$ defined by any tuple of C_{b_i} that fulfills (6.57). In the example this can be seen noting $C_{b_i}(\gamma_m) = C_{b_i}^1 = M_b^*$ and that (6.29) is equal to (6.58), i.e.

$$\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_i}^1\rangle) = \mathcal{I}_{s}^{\mathbb{M}}(\gamma_m) \tag{6.61}$$

To summarize the discussion above, the least optimistic structural isolability is obtained with the smallest set C_b for each system behavioral mode b that can be proven to fulfill (6.57). Calculate the primitive structural isolability $\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_b\rangle)$ to improve the structural isolability. It holds that $\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_{sp}^{\mathbb{M}}(\langle C_b\rangle)$. Finally the primitive structural isolability obtained if $C_b = M_b^*$ is equal to the structural isolability obtained by a γ that fulfills the condition of Theorem 6.5.

In the next section the items in Theorem 6.6 will be proven. For readers that are mainly interested in using this technique, it is possible to omit the following section and continue reading Section 6.4.2.

Theoretical Results Proving Theorem 6.6

Each item except for item (c) in Theorem 6.6 will have a corresponding lemma. Item (c) is just a reformulation of the result stated in Theorem 6.5. The first item is stated and proven in the next lemma.

Lemma 6.7. *If* $\mathbb{M}$ *is a diagnostic model,* $b_1, \dots, b_n$ *are the system behavioral modes, and* $\langle C_{b_1}, \dots, C_{b_n}\rangle$ *is a tuple of one model for each behavioral mode such that* $C_{b_i} \subseteq M_{b_i}$ *then*

$$\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_1}, \dots, C_{b_n}\rangle) \subseteq \mathcal{I}_{s}^{\mathbb{M}}(\{C_{b_1}, \dots, C_{b_n}\}) \tag{6.62}$$

Proof. The right part of (6.62) can be expressed using (6.3) as

$$\mathcal{I}_{s}^{\mathbb{M}}(\{C_{b_1}, \dots, C_{b_n}\}) = \bigcup_{b_i \in \mathcal{B}} \{(b_1, b_2) | b_1 \notin \text{ass}\, C_{b_i} \wedge b_2 \in \text{ass}\, C_{b_i}\} \tag{6.63}$$

Since $C_{b_i} \subseteq M_{b_i}$, Lemma 5.3 implies that $b_i \in \text{ass}\, C_{b_i}$. Using this expression, the right-hand side of expression (6.63) can be rewritten as

$$\bigcup_{b_i \in \mathcal{B}} \left(\{(b_1, b_i) | b_1 \notin \text{ass}\, C_{b_i}\} \cup \{(b_1, b_2) | b_1 \notin \text{ass}\, C_{b_i} \wedge b_2 \in \text{ass}\, C_{b_i} \wedge b_2 \neq b_i\}\right) \tag{6.64}$$

or equivalently

$$\{(b_1, b_2)|b_1 \notin \text{ass}\, C_{b_2}\} \cup \bigcup_{b_i \in \mathcal{B}} \{(b_1, b_2)|b_1 \notin \text{ass}\, C_{b_i} \wedge b_2 \in \text{ass}\, C_{b_i} \wedge b_2 \neq b_i\} \quad (6.65)$$

Hence the structural isolability $\mathcal{I}_s^{\mathbb{M}}(\{C_{b_1}, \dots, C_{b_n}\})$ is equal to (6.65) and it follows that

$$\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_1}, \dots, C_{b_n}\rangle) \subseteq \mathcal{I}_s^{\mathbb{M}}(\{C_{b_1}, \dots, C_{b_n}\}) \quad (6.66)$$

which completes the proof. □

Next item (b) in Theorem 6.6 will be proven in the following two lemmas. Given a diagnostic model $\mathbb{M}$, let A_b denote a set defined as

$$A_b := \{(b_1, b)|b_1 \in \mathcal{B}\} \quad (6.67)$$

Lemma 6.8. *Let $\mathbb{M}$ be a diagnostic model and let γ be an arbitrary set of models. If $b_0 \in \mathcal{B}$, $C_{b_0}(\gamma)$ is defined as in (6.51), and A_{b_0} is defined as in (6.67), then*

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) \cap A_{b_0} = \mathcal{I}_s^{\mathbb{M}}(\{C_{b_0}(\gamma)\}) \cap A_{b_0} \quad (6.68)$$

Before Lemma 6.8 is proven an example of the result (6.68) is presented.

Example 6.6 Consider the diagnostic model in Example 6.1. Let

$$\gamma := \{\{e_5\}, \{e_1, e_2\}\} \quad (6.69)$$

and $b_0 := b_4$ in (6.68). Then it follows that

$$\begin{aligned} C_{b_4}(\gamma) &= \{e_1, e_2, e_5\} \\ A_{b_4} &= \{(b_i, b_4)|b_i \in \{b_1, \dots, b_8\}\} \end{aligned} \quad (6.70)$$

Remember that $\mathcal{I}_s^{\mathbb{M}}(\gamma)$ is calculated as

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) = \Big(\bigcup_{M \in \gamma} \{(b_i, b_j)|b_i \notin \text{ass}\, M \wedge b_j \in \text{ass}\, M\}\Big) \quad (6.71)$$

and using (6.67) it follows that

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) \cap A_{b_4} = \Big(\bigcup_{M \in \gamma} \{(b_i, b_4)|b_i \notin \text{ass}\, M \wedge b_4 \in \text{ass}\, M\}\Big)$$

Using that $\text{ass}\{e_5\} = \{b_4, b_6, b_7, b_8\}$ and $\text{ass}\{e_1, e_2\} = \{b_1, b_3, b_4, b_7\}$ imply that $b_4 \in \text{ass}\{e_5\}$ and $b_4 \in \text{ass}\{e_1, e_2\}$ respectively. It follows that

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) \cap A_{b_4} = \{(b_i, b_4)|b_i \in \{b_1, b_2, b_3, b_5\}\} \cup \{(b_i, b_4)|b_i \in \{b_2, b_5, b_6, b_8\}\}$$

Then

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) \cap A_{b_4} = \{(b_i, b_4)|b_i \in \{b_1, b_2, b_3, b_5, b_6, b_8\}\} \quad (6.72)$$

The right-hand side of (6.68) is in this example

$$\mathcal{I}_s^{\mathbb{M}}(\{C_{b_4}(\gamma)\}) \cap A_{b_4} = \{(b_i, b_4) | b_i \in \{b_1, b_2, b_3, b_5, b_6, b_8\}\} \tag{6.73}$$

because $\text{ass}\, C_{b_4}(\gamma) = \text{ass}\{e_1, e_2, e_5\} = \{b_4, b_7\}$. Equation (6.72) and (6.73) are equal as Lemma 6.8 stated.

Proof. From (6.3) it follows that

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) \cap A_{b_0} = \Big(\bigcup_{M \in \gamma} \{(b_1, b_2) | b_1 \notin \text{ass}\, M \wedge b_2 \in \text{ass}\, M\}\Big) \cap A_{b_0} \tag{6.74}$$

Using (6.67) it follows that (6.74) can be expressed as

$$\Big(\bigcup_{M \in \gamma} \{(b_1, b_2) | b_1 \notin \text{ass}\, M \wedge b_2 \in \text{ass}\, M\}\Big) \cap A_{b_0} = \bigcup_{M \in \gamma} \{(b_1, b_0) | b_1 \notin \text{ass}\, M \wedge b_0 \in \text{ass}\, M\} \tag{6.75}$$

From Lemma 5.3 it follows that

$$\bigcup_{M \in \gamma} \{(b_1, b_0) | b_1 \notin \text{ass}\, M \wedge b_0 \in \text{ass}\, M\} = \bigcup_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} \{(b_1, b_0) | b_1 \notin \text{ass}\, M\} \tag{6.76}$$

Using basic set theory it holds that

$$\bigcup_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} \{(b_1, b_0) | b_1 \notin \text{ass}\, M\} = \bigcup_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} \{(b_1, b_0) | b_1 \in (\text{ass}\, M)^C\} \tag{6.77}$$

Since b_0 is fixed it follows that

$$\bigcup_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} \{(b_1, b_0) | b_1 \in (\text{ass}\, M)^C\} = \{(b_1, b_0) | b_1 \in \bigcup_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} (\text{ass}\, M)^C\} \tag{6.78}$$

Set theory gives that

$$\{(b_1, b_0) | b_1 \in \bigcup_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} (\text{ass}\, M)^C\} = \{(b_1, b_0) | b_1 \in \Big(\bigcap_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} \text{ass}\, M\Big)^C\} \tag{6.79}$$

The definition of operator ass implies that

$$\{(b_1, b_0) | b_1 \in \Big(\bigcap_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} \text{ass}\, M\Big)^C\} = \{(b_1, b_0) | b_1 \in \Big(\bigcap_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} (\bigcap_{e \in M} \text{ass}\, e)\Big)^C\} \tag{6.80}$$

It is equivalent to intersect over all equations contained in some model $M \in \gamma$ such that $M \subseteq M_{b_0}$. From the definition of $C_{b_0}(\gamma)$ in (6.51) it follows that

$$\{(b_1, b_0) | b_1 \in \Big(\bigcap_{\substack{M \in \gamma \\ M \subseteq M_{b_0}}} (\bigcap_{e \in M} \text{ass}\, e)\Big)^C\} = \{(b_1, b_0) | b_1 \in \Big(\bigcap_{e \in C_{b_0}(\gamma)} \text{ass}\, e\Big)^C\} \tag{6.81}$$

From the definition of operator ass it follows that

$$\{(b_1, b_0) | b_1 \in \Big(\bigcap_{e \in C_{b_0}(\gamma)} \operatorname{ass} e\Big)^C\} = \{(b_1, b_0) | b_1 \in \big(\operatorname{ass} C_{b_0}(\gamma)\big)^C\} \tag{6.82}$$

Basic set theory implies

$$\{(b_1, b_0) | b_1 \in \big(\operatorname{ass} C_{b_0}(\gamma)\big)^C\} = \{(b_1, b_0) | b_1 \notin \operatorname{ass} C_{b_0}(\gamma)\} \tag{6.83}$$

The right-hand side of (6.68) can be written as

$$\mathcal{I}_s^{\mathbb{M}}(\{C_{b_0}(\gamma)\}) \cap A_{b_0} = \{(b_1, b_2) | b_1 \notin \operatorname{ass} C_{b_0}(\gamma) \land b_2 \in \operatorname{ass} C_{b_0}(\gamma)\} \cap A_{b_0} \tag{6.84}$$

Using the definition of A_{b_0} it follows that

$$\begin{aligned} &\{(b_1, b_2) | b_1 \notin \operatorname{ass} C_{b_0}(\gamma) \land b_2 \in \operatorname{ass} C_{b_0}(\gamma)\} \cap A_{b_0} = \\ &\{(b_1, b_0) | b_1 \notin \operatorname{ass} C_{b_0}(\gamma) \land b_0 \in \operatorname{ass} C_{b_0}(\gamma)\} \end{aligned} \tag{6.85}$$

From $C_{b_0}(\gamma) \subseteq M_{b_0}$, and Lemma 5.3 it follows that

$$\{(b_1, b_0) | b_1 \notin \operatorname{ass} C_{b_0}(\gamma) \land b_0 \in \operatorname{ass} C_{b_0}(\gamma)\} = \{(b_1, b_0) | b_1 \notin \operatorname{ass} C_{b_0}(\gamma)\} \tag{6.86}$$

Since that the right-hand side of (6.83) and (6.86) are equal Lemma 6.8 follows. □

Lemma 6.9. *If* $\mathbb{M}$ *is a diagnostic model,* γ *is an arbitrary set of models, and* $C_b(\gamma)$ *is defined as in (6.51), then*

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) = \mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_1}(\gamma), \ldots, C_{b_n}(\gamma)\rangle) \tag{6.87}$$

Expression (6.87) can be used to calculate $\mathcal{I}_s^{\mathbb{M}}(\gamma)$ in a different way than in Algorithm 6.1. According to expression (6.87), $\mathcal{I}_s^{\mathbb{M}}(\gamma)$ can be calculated by calculating C_b and then $\operatorname{ass} C_b$for each system behavioral mode b. However Algorithm 6.1 is less computational intense. Hence Lemma 6.9 is mainly important for theoretical purposes.

Proof. From (6.67) it follows that

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) = \bigcup_{b_j \in \mathcal{B}} (\mathcal{I}_s^{\mathbb{M}}(\gamma) \cap A_{b_j}) \tag{6.88}$$

Then from the equalities (6.74)-(6.83) it follows that

$$\bigcup_{b_j \in \mathcal{B}} (\mathcal{I}_s^{\mathbb{M}}(\gamma) \cap A_{b_j}) = \bigcup_{b_j \in \mathcal{B}} \{(b_i, b_j) | b_i \notin \operatorname{ass} C_{b_j}(\gamma)\} \tag{6.89}$$

Finally it holds that

$$\bigcup_{b_j \in \mathcal{B}} \{(b_i, b_j) | b_i \notin \operatorname{ass} C_{b_j}(\gamma)\} = \{(b_i, b_j) | b_i \notin \operatorname{ass} C_{b_j}(\gamma)\} \tag{6.90}$$

which can be rewritten using (6.3) and (6.52) as

$$\mathcal{I}_s^{\mathbb{M}}(\gamma) = \mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_1}(\gamma), \ldots, C_{b_n}(\gamma)\rangle) \tag{6.91}$$

□

Item (d) in Theorem 6.6 is proven in the following Lemma.

Lemma 6.10. *Let* $\mathbb{M}$ *be a diagnostic model, let* $b_1, \ldots, b_n$ *be the system behavioral modes, and let* $(C_{b_1}, \ldots, C_{b_n})$ *be a tuple of one model for each system behavioral mode. If*

$$\forall b_i \in \mathcal{B} : C_{b_i} \subseteq C'_{b_i} \tag{6.92}$$

then

$$\mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_1}(\gamma), \ldots, C_{b_n}(\gamma)\rangle) \subseteq \mathcal{I}_{sp}^{\mathbb{M}}(\langle C'_{b_1}(\gamma), \ldots, C'_{b_n}(\gamma)\rangle) \tag{6.93}$$

Proof. Take an arbitrary $b \in \mathcal{B}$ then it holds that

$$C_b \subseteq C'_b \tag{6.94}$$

From (6.94) and the definition of ass it follows that

$$\text{ass}\, C'_b \subseteq \text{ass}\, C_b \tag{6.95}$$

Expression (6.95) is equivalent to

$$\mathcal{B} \backslash \text{ass}\, C_b \subseteq \mathcal{B} \backslash \text{ass}\, C'_b \tag{6.96}$$

Expression (6.96) implies that

$$\{(b_i, b) | b_i \in \mathcal{B} \backslash \text{ass}\, C_b\} \subseteq \{(b_i, b) | b_i \in \mathcal{B} \backslash \text{ass}\, C'_b\} \tag{6.97}$$

Expression (6.97) is equivalent to

$$\{(b_i, b) | b_i \notin \text{ass}\, C_b\} \subseteq \{(b_i, b) | b_i \notin \text{ass}\, C'_b\} \tag{6.98}$$

Since b was arbitrarily chosen it follows that

$$\bigcup_{b \in \mathcal{B}} \{(b_i, b) | b_i \notin \text{ass}\, C_b\} \subseteq \bigcup_{b \in \mathcal{B}} \{(b_i, b) | b_i \notin \text{ass}\, C'_b\} \tag{6.99}$$

Hence it holds that

$$\{(b_i, b_j) | b_i \notin \text{ass}\, C_{b_j}\} \subseteq \{(b_i, b_j) | b_i \notin \text{ass}\, C'_{b_j}\} \tag{6.100}$$

Rewriting (6.100) using (6.52) completes the proof. □

Item (e) in Theorem 6.6 is proven in the following Lemma.

Lemma 6.11. *Let* $\mathbb{M}$ *be a diagnostic model that fulfills Assumption 6.1, let* $b_1, \ldots, b_n$ *be the system behavioral modes, and let* $(C_{b_1}, \ldots, C_{b_n})$ *be a tuple of one model for each behavioral mode such that* $C_{b_i} \subseteq M_{b_i}$. *If*

$$\forall b_i \in \mathcal{B} : M^*_{b_i} \subseteq C_{b_i} \tag{6.101}$$

then there exists a γ *such that*

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma) \subseteq \mathcal{I}_{sp}^{\mathbb{M}}(\langle C_{b_1}, \ldots, C_{b_n}\rangle) \tag{6.102}$$

Proof. For any diagnostic model $\mathbb{M}$ there exists a unique set of minimal rejectable models γ_m. Let $\gamma_b := \{M \in \gamma_m | M \subseteq M_b\}$. According to the definition of Σ_b it follows that models in Σ_b are the maximal models that are included in γ_b. Since this holds for all system behavioral modes it follows that γ_m satisfy (6.34). Since $\mathbb{M}$ fulfills Assumption 6.1, it follows from Theorem 6.5 that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma_b) \tag{6.103}$$

From Lemma 6.9 it follows that

$$\mathcal{I}_s^{\mathbb{M}}(\gamma_b) = \{(b_i, b_j) | b_i \notin \text{ass}\, C_{b_j}(\gamma_b)\} \tag{6.104}$$

Since γ_b fulfills condition (6.34), then it follows that

$$\forall b \in \mathcal{B} : C_b(\gamma_b) = M_b^* \tag{6.105}$$

Now for any tuple $(C_{b_1}, \ldots, C_{b_n})$ such that (6.101) is fulfilled it follows from (6.105) that

$$\forall b_i \in \mathcal{B} : C_{b_i}(\gamma_b) \subseteq C_{b_i} \tag{6.106}$$

Since (6.106) is equivalent with (6.92) the condition in Lemma 6.10 is fulfilled and it follows that

$$\{(b_i, b_j) | b_i \notin \text{ass}\, C_{b_j}(\gamma_b)\} \subseteq \{(b_i, b_j) | b_i \notin \text{ass}\, C_{b_j}\} \tag{6.107}$$

The conclusion (6.102) follows from (6.103), (6.104), (6.107), and (6.52). □

6.4.2 System Behavioral Models

Depending on available knowledge contained in the diagnostic model, the smallest set C_b will be chosen such that $M_b^* \subseteq C_b$ can validated for each system behavioral mode to obtain the least optimistic structural isolability given that level of knowledge. Two different approaches that calculate the best structural isolability given different levels of information of the diagnostic model are presented in this and in the next section respectively.

To ensure that $M_b^* \subseteq C_b$ holds if no analytical properties are known, the only option is to use $C_b = M_b$ for all system behavioral modes. This choice can also be written as $\gamma = \gamma_B$. From item (e) in Theorem 6.6 it follows that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma_B) \tag{6.108}$$

It is not likely that item (c) in Theorem 6.6 is fulfilled and if no analytical properties are known, it is impossible to check if item (c) in Theorem 6.6 is fulfilled. An example of the resulting structural isolability using γ_B is shown in (6.60). Remember also that the structural isolability obtained from γ_B in (6.60) is not as good as the structural isolability obtained using γ_m in (6.58). In general, the structural isolability calculated using γ_B is optimistic.

6.4.3 Structurally Over-determined Models

In previous section no analytical information was available and therefore the C_b:s could not be chosen smaller than M_b to be sure that $M_b^* \subseteq C_b$ holds. In this section we assume that analytical expressions are available and it will be shown how the sets C_b can be chosen smaller then M_b. Then according to item (d) in Theorem 6.6, the resulting structural isolability is better than the one obtained using γ_B.

The method that will be presented uses first the structure of the model to suggest a tuple of C_b:s. Then each of these models are analyzed, first structurally and if needed also analytically to validate that $M_b^* \subseteq C_b$ holds.

The idea is that the structure of the model suggests a small C_b that is likely to have the property $M_b^* \subseteq C_b$. For the structurally overdetermined part M_b^+ defined in Section 3.4.2, it holds for example in the linear case that $M_b^* \subseteq M_b^+$ if the matrix, corresponding to $M_b \setminus M_b^+$ and the unknown variables not in M_b^+, has full row-rank. This could make it a reasonable choice to find M_b^+ for all system behavioral modes b.

To make this method applicable to models that are not linear, results to decide if $M_b^* \subseteq C_b$ is fulfilled have to be given. The next theorem gives sufficient conditions for a model $\hat{M} \subseteq M_b$ to fulfill $M_b^* \subseteq \hat{M}$.

Theorem 6.12. *Let M_b and $\hat{M} \subseteq M_b$ be two models and let (6.32) be fulfilled for b. If $\hat{X} := \mathrm{var}_{X_u} \hat{M}$, $\bar{X} := X_u \setminus \hat{X}$, $\bar{M} := M_b \setminus \hat{M}$, and*

$$\forall \mathbf{z} \forall \hat{\mathbf{x}} : \big(\hat{M}(\mathbf{z}, \hat{\mathbf{x}}) \rightarrow \exists \bar{\mathbf{x}} : \bar{M}(\mathbf{z}, \hat{\mathbf{x}}, \bar{\mathbf{x}})\big) \tag{6.109}$$

then it follows that $M_b^* \subseteq \hat{M}$, where M_b^* is defined in (6.33).

Proof. From (6.109) and the definition of $\bar{M}$ it follows that

$$\forall \mathbf{z} \forall \hat{\mathbf{x}} : \big(\hat{M}(\mathbf{z}, \hat{\mathbf{x}}) \rightarrow \exists \bar{\mathbf{x}} : M_b(\mathbf{z}, \hat{\mathbf{x}}, \bar{\mathbf{x}})\big) \tag{6.110}$$

This means for an arbitrary $\hat{\mathbf{x}}_1$ that

$$\forall \mathbf{z} : \big(\hat{M}(\mathbf{z}, \hat{\mathbf{x}}_1) \rightarrow \exists \bar{\mathbf{x}} : M_b(\mathbf{z}, \hat{\mathbf{x}}_1, \bar{\mathbf{x}})\big) \tag{6.111}$$

Then it follows that

$$\forall \mathbf{z} : \big(\exists \hat{\mathbf{x}} : \hat{M}(\mathbf{z}, \hat{\mathbf{x}}) \rightarrow \exists \hat{\mathbf{x}} \exists \bar{\mathbf{x}} : M_b(\mathbf{z}, \hat{\mathbf{x}}, \bar{\mathbf{x}})\big) \tag{6.112}$$

From the definition of M_b^* and from (6.32) it follows that M_b^* is the unique minimal set of equations M such that

$$\{\mathbf{z} | \exists \mathbf{x} M(\mathbf{z}, \mathbf{x})\} = \{\mathbf{z} | \exists \mathbf{x} M_b(\mathbf{z}, \mathbf{x})\} \tag{6.113}$$

holds. Since $\hat{M}$ fulfills (6.113) according to (6.112), it follows from the fact that M_b^* is the unique minimal set of equations that satisfies (6.113) that

$$M_b^* \subseteq \hat{M} \tag{6.114}$$

□

Next a weaker alternative to Theorem 6.12 is presented. The advantage is that the next condition requires less computations to be validated.

Corollary 6.13. *Let* M_b *and* $\hat{M} \subseteq M_b$ *be two models and let (6.32) be fulfilled for* b*. If* $\hat{X} := \text{var}_{X_u} \hat{M}$*,* $\bar{X} := X_u \setminus \hat{X}$*,* $\bar{M} := M_b \setminus \hat{M}$*, and*

$$\forall \mathbf{z} \forall \hat{\mathbf{x}} \exists \bar{\mathbf{x}} : \bar{M}(\mathbf{z}, \hat{\mathbf{x}}, \bar{\mathbf{x}}) \tag{6.115}$$

then it follows that $M_b^* \subseteq \hat{M}$ *where* M_b^* *is defined in (6.33).*

Corollary 6.13 follows directly from (6.109) in Theorem 6.12. Note that in Corollary 6.13 the analytical test (6.115) only involves the equations to be removed from M_b. Since not all equations are analyzed, the computations become easier. An example will show how a model is tested according to Theorem 6.12 and Corollary 6.13.

Example 6.7 Consider the diagnostic model in Example 6.1. Assume that we want to check if $M_{b_6}^* \subseteq \{e_5\}$. The models defined in Theorem 6.12 are in this example $M_{b_6} = \{e_2, e_3, e_5\}$, $\hat{M} = \{e_5\}$, and $\bar{M} = \{e_2, e_3\}$. Even if the model $M_{b_6}^*$ is shown in previous examples, $M_{b_6}^*$ is in this example considered to be unknown. The set of variables in Theorem 6.12 are $\hat{X} = \varnothing$ and $\bar{X} = \{x_1, x_2\}$. To validate (6.109) the set

$$\{(\mathbf{z}, \hat{\mathbf{x}}) | \hat{M}(\mathbf{z}, \hat{\mathbf{x}})\} \tag{6.116}$$

is calculated. In this example it is

$$\{\mathbf{z} | z_3 = 0\} \tag{6.117}$$

Now, (6.109) holds if

$$\forall \mathbf{z} \in \{\mathbf{z} | z_3 = 0\} \exists x_1, x_2 : \{e_2, e_3\} \tag{6.118}$$

or equivalently

$$\forall z_2 \exists x_1, x_2 : \{x_1 = x_2^2, z_2 = x_2\} \tag{6.119}$$

For an arbitrary z_2, x_2 is defined as $x_2 := z_2$ and then x_1 is defined as $x_1 := x_2^2 = z_2^2$. This assignment prove that (6.119) holds. Hence according to Theorem 6.12 it follows that $M_{b_6}^* \subseteq \{e_5\}$.

If $M_{b_6}^* \subseteq \{e_5\}$ is validated using Corollary 6.13 instead of Theorem 6.12, the test in this example is to validate only expression (6.119). Hence when using Corollary 6.13 the set in (6.116) needs not to be calculated.

In Section 3.4.2 it was explained how the structurally overdetermined part is found and Theorem 6.12 describes a method to test if the structural isolability is a necessary condition for the analytical isolability. Next we combine the structural algorithm from Section 3.4.2 with the suggested test such that the calculated structural isolability is a necessary condition for the analytical isolability of the diagnostic model considered.

Given a set $\{M^+_b | b \in \mathcal{B}\}$ let the set $\mathcal{B}' \subseteq \mathcal{B}$ be the system behavioral modes b such that the expression $M^*_b \subseteq M^+_b$ can be validated, for example with Theorem 6.12 or Corollary 6.13. Then for $b \in \mathcal{B}'$, C_b is preferably defined as

$$C_b := M^+_b \tag{6.120}$$

and for $b \in \mathcal{B}\backslash\mathcal{B}'$, C_b is defined as

$$C_b := M_b \tag{6.121}$$

Then from Theorem 6.6 it follows that this choice of C_b corresponds to a better structural isolability than if all C_b:s are chosen to be equal to M_b. However it is not sure $C_b = M^*_b$ holds. Therefore it is possible that there are other C_b:s that corresponds to better structural isolabilities. According to item (a) in Theorem 6.6, it holds that the best structural isolability given the chosen set of C_b:s is obtained if the primitive structural isolability $\mathcal{I}^{\mathbb{M}}_{sp}(\langle C_{b_1}, \ldots, C_{b_n}\rangle)$ is used, The primitive structural isolability is calculated as

$$\mathcal{I}^{\mathbb{M}}_{sp}(\langle C_{b_1}, \ldots, C_{b_n}\rangle) = \{(b_i, b_j) | b_i \notin \text{ass}\, C_{b_j}\} \tag{6.122}$$

Item (e) in Theorem 6.6 implies that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}^{\mathbb{M}}_{sp}(\langle C_{b_1}, \ldots, C_{b_n}\rangle) \tag{6.123}$$

An example will show how this method is applied a diagnostic model.

Example 6.8 Consider the diagnostic model in Example 6.1. The structure of the diagnostic model (6.16) is

component	assumption	equation	x_1	x_2	z_1	z_2	z_3
Sensor 1	$\phi(s_1 = NF)$	e_1 :	X		X		
Comp	$\mathcal{B}$	e_2 :	X	X			
Sensor 2	$\phi(s_2 = NF)$	e_3 :		X		X	
Sensor 3	$\phi(s_3 = NF)$	e_4 :		X			X
	$\phi(s_3 = SG)$	e_5 :					X

(6.124)

The models $M^*_{b_i}$, the structural overdetermined models $M^+_{b_i}$, and the behavioral models M_{b_i} are

b	$M^*_{b_i}$	$M^+_{b_i}$	M_{b_i}
b_1	$\{e_1, e_2, e_3, e_4\}$	$\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}\}$	$\{e_1, e_2, e_3, e_4\}$
b_2	$\{e_2, e_3, e_4\}$	$\{\mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}\}$	$\{e_2, e_3, e_4\}$
b_3	$\{e_1, e_2, e_4\}$	$\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_4}\}$	$\{e_1, e_2, e_4\}$
b_4	$\{e_1, e_2, e_3, e_5\}$	$\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_5}\}$	$\{e_1, e_2, e_3, e_5\}$
b_5	$\varnothing$	$\varnothing$	$\{e_2, e_4\}$
b_6	$\{e_5\}$	$\{\mathbf{e_5}\}$	$\{e_2, e_3, e_5\}$
b_7	$\{e_1, e_2, e_5\}$	$\{e_5\}$	$\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_5}\}$
b_8	$\{e_5\}$	$\{\mathbf{e_5}\}$	$\{e_2, e_5\}$

(6.125)

The meaning of the models that are written in bold will be explained later. The models $M^*_{b_i}$ are in this example assumed to be unknown. However since they corresponds to the best structural isolability of all C_b:s defined in Theorem 6.6, they can be of interest to show for comparison.

Each $M^+_{b_i}$ is tested to conclude if $M^*_b \subseteq M^+_{b_i}$ holds. For the behavioral modes $b_1, \ldots, b_4$ it holds that $M^+_{b_i} = M_{b_i}$ and since $M^*_{b_i} \subseteq M_{b_i}$ it follows that $M^*_{b_i} \subseteq M^+_{b_i}$ without doing any analytical calculations. If Corollary 6.13 is applied to $M^+_{b_5}$ condition (6.109) becomes

$$\forall z_3 \exists x_1, x_2 : \{x_1 = x_2^2, z_3 = x_2\} \tag{6.126}$$

This expression is easily validated finding the assignment $x_2 := z_3$ and $x_1 := x_2^2 = z_3^2$. Continuing in this way, the sets to be analyzed using Corollary 6.13 are $\bar{M} = \{e_2, e_3\}$, $\bar{M} = \{e_1, e_2\}$ and $\bar{M} = \{e_2\}$, for each remaining behavioral mode respectively.

The behavioral mode b_7 is of special interest, because it can be seen in (6.125) that $M^*_{b_7} \not\subseteq M^+_{b_7}$. First it holds that $\{e_1, e_2\}$ and $\{e_5\}$ have no variable in common. This implies that the condition in Theorem 6.12 and the condition in Corollary 6.13 are equivalent. The condition is

$$\forall z_1 \exists x_1, x_2 : \{z_1 = x_1, x_1 = x_2^2\} \tag{6.127}$$

If it is assumed that all variables are real, it is easy to verify that (6.127) is false because there is no x_2 for $z_1 < 0$. Only for behavioral mode b_7, it is not possible to verify that $M^*_{b_i} \subseteq M^+_{b_i}$. Hence $\mathcal{B}' = \mathcal{B} \backslash \{b_7\}$ and from (6.120) and (6.121) the model C_b for each behavioral mode is defined. The models C_b are written in bold in (6.125). In this example the set of C_b:s corresponds to the best structural isolability shown in (6.29) or (6.58). This can also be realized noting in (6.125) that $M^*_b = C_b$ for all system behavioral modes.

An interesting insight that is one justification for the contents of Section 6.4 is that the same structural isolability is obtained in Example 6.8 as in Example 6.3. The important difference is how the structural isolability was calculated. In Example 6.3 all minimal rejectable models where used to get the structural isolability. Detailed analytical knowledge about the diagnostic model must be used to derive all minimal rejectable models. In Example 6.8 considerable less analytical calculations were carried out to obtain the same result.

6.5 Calculating Analytical Isolability of Diagnostic Models

In this section the analytical isolability $\mathcal{I}^{\mathbb{M}}$ of the diagnostic model in Example 6.1 is calculated. The calculations show how the structural isolability can be used to reduce the amount of analytical computations significantly. The method that will be applied to the diagnostic model in Example 6.1 is not intended to hold generally. However in the construction of a general algorithm, these ideas can be used.

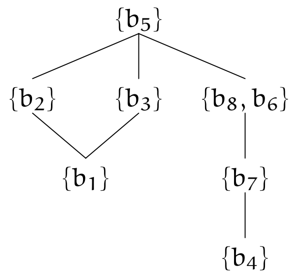

Figure 6.3 *The Hasse diagram of the partial order* $\mathcal{P}_s^{\mathbb{M}}(\gamma_m)$.

6.5.1 Calculating Analytical Isolability Using Structural Isolability

The calculations done later assumes that the structural isolability is derived from a γ such that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma) \tag{6.128}$$

A sufficient condition for γ is that Theorem 5.6 is satisfied. Then the needed property (6.128) follows from Theorem 6.2. From (5.46) in Example 5.2 the minimal rejectable models are

$$\gamma_m = \{\{e_5\}, \{e_1, e_2\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\} \tag{6.129}$$

This set satisfies the conditions of Theorem 6.2. Moreover, γ_m also satisfies Theorem 6.5 which is not needed but implies that the structural isolability is less optimistic, i.e. more similar to the analytical isolability. The structural isolability $\mathcal{I}_s^{\mathbb{M}}(\gamma_m)$ is shown in (6.29) and the partial order $\mathcal{P}_s^{\mathbb{M}}(\gamma_m)$ is shown in Figure 6.2. Since Figure 6.2 will be used a lot in the calculations, it is duplicated and is also shown in Figure 6.3.

The structural isolability $\mathcal{I}_s^{\mathbb{M}}(\gamma_m)$ can be used to calculate $\mathcal{I}^{\mathbb{M}}$ in a more efficient way, because we know that

$$\mathcal{I}^{\mathbb{M}} \subseteq \mathcal{I}_s^{\mathbb{M}}(\gamma_m) \tag{6.130}$$

It means that for those elements that satisfy

$$(b_1, b_2) \notin \mathcal{I}_s^{\mathbb{M}}(\gamma_m) \tag{6.131}$$

it follows from (6.130) that

$$(b_1, b_2) \notin \mathcal{I}^{\mathbb{M}} \tag{6.132}$$

This situation can be illustrated in the isolability matrix $I^{\mathbb{M}}$ as

$$\begin{array}{l|ccccccc} \text{present} & \multicolumn{7}{c}{\text{necessary interpreted mode}} \\ \text{mode} & \{b_1\} & \{b_2\} & \{b_3\} & \{b_4\} & \{b_7\} & \{b_8, b_6\} & \{b_5\} \\ \hline \{b_1\} & X & X & X & ? & ? & ? & X \\ \{b_2\} & ? & X & ? & ? & ? & ? & X \\ \{b_3\} & ? & ? & X & ? & ? & ? & X \\ \{b_4\} & ? & ? & ? & X & X & X & X \\ \{b_7\} & ? & ? & ? & ? & X & X & X \\ \{b_8, b_6\} & ? & ? & ? & ? & ? & X & X \\ \{b_5\} & ? & ? & ? & ? & ? & ? & X \end{array} \tag{6.133}$$

Here the "X" denotes that $(b_i, b_j) \notin \mathcal{I}^{\mathbb{M}}$ as usual, "?" denotes that we do not know if $(b_i, b_j) \notin \mathcal{I}^{\mathbb{M}}$ or $(b_i, b_j) \in \mathcal{I}^{\mathbb{M}}$ is true, and later we will use "0" to denote that $(b_i, b_j) \in \mathcal{I}^{\mathbb{M}}$. Hence it is sufficient to check the properties corresponding to the matrix elements "?", i.e. to check if

$$(b_1, b_2) \in \mathcal{I}^{\mathbb{M}} \tag{6.134}$$

holds for the isolability properties

$$(b_1, b_2) \in \mathcal{I}_s^{\mathbb{M}} \tag{6.135}$$

Remember that an equivalent expression for (6.134) is

$$\mathcal{O}_{M_{b_1}} \setminus \mathcal{O}_{M_{b_2}} \neq \varnothing \tag{6.136}$$

Since there is no risk for confusion the specification of diagnostic model is omitted in (6.136). If (6.136) is evaluated for every ordered pair of two different system-behavioral modes, then 48 evaluations have to be done. If only ordered pairs of equivalent classes included in the structural isolability (6.29) are evaluated, the number of evaluations is equal to the number of "?" in (6.133), i.e. 31. However using the partial order $\mathcal{P}_s^{\mathbb{M}}(\gamma_m)$ it is even not necessary to evaluate all 31 pairs of equivalent classes, it will be shown that only 7 evaluations are sufficient to calculate the analytical isolability.

6.5.2 Minimal and Maximal Elements of Partial Orders

Before we continue some notions concerning partial orders are defined. If R is a partial order on A, then an element $a \in A$ is called a *minimal* element of A if for all $x \in A$ and $x \neq a$ implies that $(a, x) \notin R$. An element b is called a *maximal* element of A if whenever $y \in A$ and $y \neq b$, then $(y, b) \notin R$.

6.5.3 Method Description

The method considers one system behavioral mode at a time. We find a minimal element, a, of $\mathcal{P}_s^{\mathbb{M}}$ that is not already considered. For each maximal element among those elements b such that there is a "?" in the entry (a, b) in $I^{\mathbb{M}}$ check if $(a, b) \in \mathcal{I}^{\mathbb{M}}$ by validating (6.136). Then update the matrix $I^{\mathbb{M}}$ by setting "0":s in all entries (b_i, b_j) such that

$$([a], [b_i]) \in \mathcal{P}_s^{\mathbb{M}}(\gamma_m) \tag{6.137}$$

and

$$([b_j], [b]) \in \mathcal{P}_s^{\mathbb{M}}(\gamma_m) \tag{6.138}$$

Next it will be proven for any such pair (b_i, b_j), that $(a, b) \in \mathcal{I}^{\mathbb{M}}$ implies $(b_i, b_j) \in \mathcal{I}^{\mathbb{M}}$ or equivalently $\mathcal{O}_{M_{b_i}} \backslash \mathcal{O}_{M_{b_j}} \neq \emptyset$. Take an arbitrary pair of behavioral modes (b_i, b_j) that fulfills (6.137) and (6.138). From the definition of $\mathcal{P}_s^{\mathbb{M}}(\gamma_m)$ it follows that for any b_1 and b_2 it holds that

$$([b_1], [b_2]) \in \mathcal{P}_s^{\mathbb{M}}(\gamma_m) \rightarrow \mathcal{O}_{M_{b_1}} \subseteq \mathcal{O}_{M_{b_2}} \tag{6.139}$$

Applying this expression to (6.137) and (6.138) implies that

$$\mathcal{O}_{M_a} \subseteq \mathcal{O}_{M_{b_i}} \tag{6.140}$$

and

$$\mathcal{O}_{M_{b_j}} \subseteq \mathcal{O}_{M_b} \tag{6.141}$$

respectively. Using that

$$\mathcal{O}_{M_a} \backslash \mathcal{O}_{M_b} \neq \emptyset \tag{6.142}$$

(6.140), (6.141), and elementary rules concerning sets gives that

$$\mathcal{O}_{M_{b_i}} \backslash \mathcal{O}_{M_{b_j}} \neq \emptyset \tag{6.143}$$

which was the expression to be proven.

6.5.4 Calculation of the Analytical Isolability

Now we return to the calculation of the analytical isolability of the diagnostic model in Example 6.1. The sets of consistent observations for each behavioral mode respectively are

b	$\mathcal{O}_{M_b}$
b_1	$\{\mathbf{z} \mid z_1 = z_2^2, z_2 = z_3\}$
b_2	$\{\mathbf{z} \mid z_2 = z_3\}$
b_3	$\{\mathbf{z} \mid z_1 = z_3^2\}$
b_4	$\{\mathbf{z} \mid z_1 = z_2^2, z_3 = 0\}$
b_5	$\mathbb{R}^3$
b_6	$\{\mathbf{z} \mid z_3 = 0\}$
b_7	$\{\mathbf{z} \mid z_1 \geq 0, z_3 = 0\}$
b_8	$\{\mathbf{z} \mid z_3 = 0\}$

(6.144)

The minimal elements of $\mathcal{P}_s^{\mathbb{M}}$ in Figure 6.3 is $\{b_1\}$ and $\{b_4\}$. We start to investigate $\{b_1\}$. In the row corresponding to $\{b_1\}$ in (6.133) there is a "?" in the columns corresponding to the elements $\{b_4\}$, $\{b_7\}$, $\{b_6, b_8\}$. Of these three elements, $\{b_6, b_8\}$ is the only maximal element defined by $\mathcal{P}_s^{\mathbb{M}}$ in Figure 6.3. Then we test if $(\{b_1\},\{b_6, b_8\}) \notin \mathcal{P}_s^{\mathbb{M}}$ by choosing arbitrary representatives of the equivalent classes respectively, for example b_1 and b_6. From (6.144) it is easy to see that

$$\mathcal{O}_{M_{b_1}} \setminus \mathcal{O}_{M_{b_6}} \neq \emptyset \tag{6.145}$$

is true. From $\mathcal{P}_s^{\mathbb{M}}(\gamma_m)$, (6.139), and (6.145) it follows that all of the following inequalities holds

$$\begin{array}{c}
\mathcal{O}_{M_{b_1}} \setminus \mathcal{O}_{M_{b_8}} \neq \emptyset \\
\mathcal{O}_{M_{b_1}} \setminus \mathcal{O}_{M_{b_6}} \neq \emptyset \\
\mathcal{O}_{M_{b_1}} \setminus \mathcal{O}_{M_{b_7}} \neq \emptyset \\
\mathcal{O}_{M_{b_1}} \setminus \mathcal{O}_{M_{b_4}} \neq \emptyset \\
\hline
\mathcal{O}_{M_{b_2}} \setminus \mathcal{O}_{M_{b_8}} \neq \emptyset \\
\mathcal{O}_{M_{b_2}} \setminus \mathcal{O}_{M_{b_6}} \neq \emptyset \\
\mathcal{O}_{M_{b_2}} \setminus \mathcal{O}_{M_{b_7}} \neq \emptyset \\
\mathcal{O}_{M_{b_2}} \setminus \mathcal{O}_{M_{b_4}} \neq \emptyset \\
\hline
\mathcal{O}_{M_{b_3}} \setminus \mathcal{O}_{M_{b_8}} \neq \emptyset \\
\mathcal{O}_{M_{b_3}} \setminus \mathcal{O}_{M_{b_6}} \neq \emptyset \\
\mathcal{O}_{M_{b_3}} \setminus \mathcal{O}_{M_{b_7}} \neq \emptyset \\
\mathcal{O}_{M_{b_3}} \setminus \mathcal{O}_{M_{b_4}} \neq \emptyset \\
\hline
\mathcal{O}_{M_{b_5}} \setminus \mathcal{O}_{M_{b_8}} \neq \emptyset \\
\mathcal{O}_{M_{b_5}} \setminus \mathcal{O}_{M_{b_6}} \neq \emptyset \\
\mathcal{O}_{M_{b_5}} \setminus \mathcal{O}_{M_{b_7}} \neq \emptyset \\
\mathcal{O}_{M_{b_5}} \setminus \mathcal{O}_{M_{b_4}} \neq \emptyset
\end{array} \tag{6.146}$$

Next it is explained as an example how $\mathcal{O}_{M_{b_2}} \setminus \mathcal{O}_{M_{b_4}} \neq \emptyset$ in (6.146) is implied by (6.145). In Figure 6.4 it is shown how (6.145) implies that $\mathcal{O}_{M_{b_2}} \setminus \mathcal{O}_{M_{b_4}} \neq \emptyset$. The set $\mathcal{O}_{M_{b_1}} \setminus \mathcal{O}_{M_{b_6}}$ is colored in gray. From $\mathcal{P}_s^{\mathbb{M}}(\gamma_m)$ we know that $\mathcal{O}_{M_{b_1}} \subseteq \mathcal{O}_{M_{b_2}}$ and $\mathcal{O}_{M_{b_4}} \subseteq \mathcal{O}_{M_{b_6}}$. In the figure it is easy to see that if $\mathcal{O}_{M_{b_1}} \setminus \mathcal{O}_{M_{b_6}} \neq \emptyset$ then $\mathcal{O}_{M_{b_2}} \setminus \mathcal{O}_{M_{b_4}} \neq \emptyset$.

If inequality $\mathcal{O}_{M_{b_i}} \setminus \mathcal{O}_{M_{b_j}} \neq \emptyset$ holds then $(b_i, b_j) \in \mathcal{I}^{\mathbb{M}}$. From (6.146) it follows that $I^{\mathbb{M}}$ is

present mode	necessary interpreted mode $\{b_1\}$	$\{b_2\}$	$\{b_3\}$	$\{b_4\}$	$\{b_7\}$	$\{b_8, b_6\}$	$\{b_5\}$
$\{b_1\}$	X	X	X	0	0	0	X
$\{b_2\}$	?	X	?	0	0	0	X
$\{b_3\}$	?	?	X	0	0	0	X
$\{b_4\}$	?	?	?	X	X	X	X
$\{b_7\}$	?	?	?	?	X	X	X
$\{b_8, b_6\}$	?	?	?	?	?	X	X
$\{b_5\}$	?	?	?	0	0	0	X

(6.147)

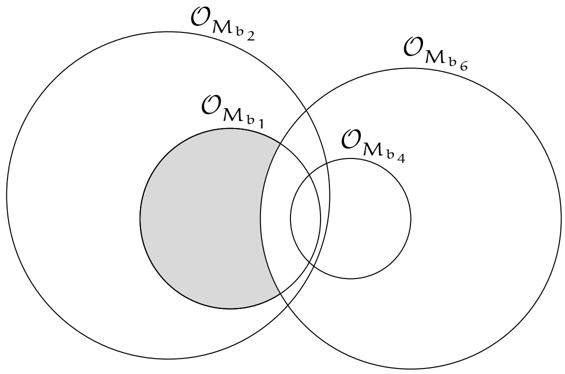

Figure 6.4 *The set* $\mathcal{O}_{M_{b_1}} \backslash \mathcal{O}_{M_{b_6}}$ *is colored in gray. The set* $\mathcal{O}_{M_{b_2}} \backslash \mathcal{O}_{M_{b_4}}$ *is a superset of* $\mathcal{O}_{M_{b_1}} \backslash \mathcal{O}_{M_{b_6}}$.

Now the element $\{b_1\}$ has been analyzed. The minimal elements of $\mathcal{P}_s^{\mathbb{M}}$ in Figure 6.3 when $\{b_1\}$ is removed are $\{b_2\}$, $\{b_3\}$, and $\{b_4\}$. Consider $\{b_2\}$. The columns with "?" are $\{b_1\}$ and $\{b_3\}$. The maximal element of these are $\{b_3\}$. Then

$$\mathcal{O}_{M_{b_2}} \backslash \mathcal{O}_{M_{b_3}} \neq \varnothing \tag{6.148}$$

is found to be true. Inequality (6.148) together with $\mathcal{P}_s^{\mathbb{M}}(\gamma_m)$ and (6.139) imply that

$$\begin{array}{c} \mathcal{O}_{M_{b_2}} \backslash \mathcal{O}_{M_{b_3}} \neq \varnothing \\ \mathcal{O}_{M_{b_5}} \backslash \mathcal{O}_{M_{b_3}} \neq \varnothing \\ \hline \mathcal{O}_{M_{b_2}} \backslash \mathcal{O}_{M_{b_1}} \neq \varnothing \\ \mathcal{O}_{M_{b_5}} \backslash \mathcal{O}_{M_{b_1}} \neq \varnothing \end{array} \tag{6.149}$$

and the updated analytical isolability matrix is

present mode	necessary interpreted mode $\{b_1\}$	$\{b_2\}$	$\{b_3\}$	$\{b_4\}$	$\{b_7\}$	$\{b_8, b_6\}$	$\{b_5\}$
$\{b_1\}$	X	X	X	0	0	0	X
$\{b_2\}$	0	X	0	0	0	0	X
$\{b_3\}$	?	?	X	0	0	0	X
$\{b_4\}$	?	?	?	X	X	X	X
$\{b_7\}$	?	?	?	?	X	X	X
$\{b_8, b_6\}$	?	?	?	?	?	X	X
$\{b_5\}$	0	?	0	0	0	0	X

(6.150)

Continuing in this way it follows that

present mode	necessary interpreted mode						
	$\{b_1\}$	$\{b_2\}$	$\{b_3\}$	$\{b_4\}$	$\{b_7\}$	$\{b_8, b_6\}$	$\{b_5\}$
$\{b_1\}$	X	X	X	0	0	0	X
$\{b_2\}$	0	X	0	0	0	0	X
$\{b_3\}$	0	0	X	0	0	0	X
$\{b_4\}$	0	0	0	X	X	X	X
$\{b_7\}$	0	0	0	0	X	X	X
$\{b_8, b_6\}$	0	0	0	0	0	X	X
$\{b_5\}$	0	0	0	0	0	0	X

(6.151)

is the analytical isolability. In this example it holds that $\mathcal{I}^{\mathbb{M}} = \mathcal{I}_s^{\mathbb{M}}(\gamma_m)$. Remember that a straightforward approach would lead to 31 inequalities to test but using the structural isolability and the corresponding partial order the number of tests are decreased to only 7 tests.

7

Computing Testable Models

In Chapter 5 it was shown which sets γ that can be used to design a sound and complete diagnostic system. In this chapter we will present methods, that mainly is structural, to obtain such γ:s. In Section 7.1 two types of γ:s that correspond to sound and complete diagnostic systems are used to present some additional desired properties of γ:s. When a γ is derived, it is sufficient to design a strong test for each model in γ to derive a sound and complete diagnostic system. The difficulty of deriving a strong test for a model will in general increase with the number of equations in the model. Minimal rejectable models are therefore especially attractive to use. Since the analysis here is assumed to be mainly structural, it will be important to know the structural properties of minimal rejectable models. In Section 7.2 structural properties for minimal rejectable models are presented. These properties are the basics for the structural algorithms defined in Section 7.3. These algorithms find γ:s such that the models in γ are small and correspond to sound and complete diagnostic systems. However these algorithms have the disadvantage that they can be computational intractable for large diagnostic models. In Section 7.4 an algorithm is presented that has a lower computational complexity than the algorithms presented in Section 7.3. The reduction of complexity is gained by removing the soundness condition. Hence the output γ corresponds to a complete but not necessarily sound diagnostic systems. Even though the output γ from this algorithm is not guaranteed to correspond to a sound diagnostic system, it often holds that the corresponding diagnostic system detects most inconsistencies.

7.1 Set of Tests to Obtain Sound and Complete Diagnostic Systems

To decide suitable tests in the diagnostic system, structural analysis can be used to compute a set of models γ. If γ fulfills Theorem 5.6, then a sound and complete diagnostic system can be designed. A sufficient condition for deriving a sound and complete diagnostic system is to design a strong test for each model $M \in \gamma$. Generally it is difficult to derive tests. However, in many cases the difficulty in deriving a test increases with the number of constraints in the model M. In these cases a reasonable additional condition on γ is that the models $M \in \gamma$ should be as small as possible to simplify the calculations to obtain a test for M. In the next two sections we discuss two choices of γ that are discussed frequently in previous chapters, i.e. the set of behavior models and the set of C_b:s. These two choices of γ are discussed in the light of the additional condition, i.e. small models are preferable in γ.

7.1.1 Tests for Behavioral Models

It is easy to obtain the behavioral models M_b using the structure. However the disadvantage is, as we have mentioned earlier, that behavioral models are the largest models that can be used to design a complete and sound diagnostic system. So the advantage of easily being obtained with the structure is of little importance compared to the big disadvantages in the analytical step when diagnostic tests are designed. If it is not possible to design tests directly for M_b smaller models are those C_b:s derived with methods described in Chapter 6.

7.1.2 Tests for the Models C_b

If structural isolability analysis of the diagnostic models $\mathbb{M}$ is done, a tuple of C_b:s is known. The set $\gamma_C = \{C_b | b \in \mathcal{B}\}$ can be used to design a sound and complete diagnostic system. It is true that $C_b \subseteq M_b$ which means that the derivation of tests can sometimes be less difficult. Since some models C_b for different behavioral modes can be equal and some models C_b can be the empty set of equations it follows that the total number of models to design tests for can be less then the number of nonempty behavioral models. However if there exists a $M \in \gamma_C$ such that no test can be derived depending on too high complexity of M, it is possible to use even more detailed structural analysis. This will be explained in the later sections. The goal is to obtain a γ with the smallest models that can be used to design a sound and complete diagnostic system. Since smaller models such as the minimal rejectable models according to Corollary 5.7 can be used to design a sound and complete diagnostic system it follows that the set of C_b:s is not particularly good choice.

7.1.3 Aim at Finding the Smallest Models

We would like to find the smallest models such that a sound and complete diagnostic system can be derived. To derive a sound and complete diagnostic system, γ has to fulfill Theorem 5.6. The smallest models that satisfy expression (5.72) are according to Corollary 5.7 the set or a subset of the minimal rejectable models. Since that goal is to design a structural method, the structure of these models will be important to know.

7.2 Structural Properties of Minimal Rejectable Models

In this section we describe structural properties of minimal rejectable models. A necessary structural property for minimal rejectable models is presented in the first section. However, it will then be shown that this structural property is not sufficiently informative to reduce the number of possible models that can be minimal rejectable models. Therefore in Section 7.2.2 a stronger necessary structural condition is presented that reduces the number of possible minimal rejectable models.

7.2.1 Connected Models

First we define a structural property for a model M.

Definition 7.1 (Connected Model). *If* M *is a model such that* $\mathcal{G}(M, \mathrm{var}_{X_u} M)$ *is connected, then* M *is a* ***connected model***.

The connection between the structural property connected models and the minimal rejectable models are stated in the next theorem.

Theorem 7.1. *A minimal rejectable model is a connected model.*

Before the proof of Theorem 7.1 is presented, an example will show connected and not connected models and there connection to minimal rejectable models.

Example 7.1 Consider the diagnostic model in Example 6.8 and especially the model $M = \{e_1, e_2, e_5\}$. The bipartite graph $\mathcal{G}(M, \mathrm{var}_{X_u} M)$ is shown in Figure 7.1. The graph $\mathcal{G}(M, \mathrm{var}_{X_u} M)$ is not connected, because e_5 have no connecting edge to any other vertices. Hence M is not a connected model and therefore not a minimal rejectable model according to Theorem 7.1. In Example 5.5, it is shown that $\{e_1, e_2\}$ and $\{e_5\}$ are minimal rejectable models. In Figure 7.1 it can be seen that both these models are connected models which once again exemplify the conclusion of Theorem 7.1.

Proof. Assume that M is a minimal rejectable and a not connected model. Since M is a not connected model it follows that there is a partition $M = M_1 \cup M_2$ such that $\mathrm{var}_{X_u} M_1 \cap \mathrm{var}_{X_u} M_2 = \emptyset$. Let the set of variables $\mathrm{var}_{X_u} M_1$ and $\mathrm{var}_{X_u} M_2$ be

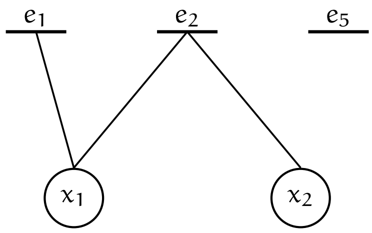

Figure 7.1 *The disconnected bipartite graph $\mathcal{G}(M, \mathrm{var}_{X_u} M)$ defined with the diagnostic model in Example 6.8 and $M = \{e_1, e_2, e_5\}$.*

denoted X_1 and X_2 respectively. Since M is a minimal rejectable model there is a $\mathbf{z}_0$ such that M is a minimal rejectable model at $\mathbf{z}_0$. Then the following calculations can be done:

$$\begin{aligned}
&\neg\exists \mathbf{x} : M(\mathbf{x}, \mathbf{z}_0) = \\
&\neg\exists \mathbf{x}_1 \exists \mathbf{x}_2 : (M_1(\mathbf{x}_1, \mathbf{z}_0) \wedge (M_2(\mathbf{x}_2, \mathbf{z}_0)) = \\
&(\neg\exists \mathbf{x}_1 : M_1(\mathbf{x}_1, \mathbf{z}_0)) \vee (\neg\exists \mathbf{x}_2 : M_2(\mathbf{x}_2, \mathbf{z}_0))
\end{aligned}$$

From the calculations above, it follows that M_1 or M_2 is also rejectable at $\mathbf{z}_0$. Since $M_1 \subset M$ and $M_2 \subset M$ this contradicts the fact that M is assumed to be a minimal rejectable model at $\mathbf{z}_0$. Since $\mathbf{z}_0$ was arbitrarily chosen such that M is a minimal rejectable model at $\mathbf{z}_0$, it follows that M is not a minimal rejectable model. □

Let a *maximal connected model* be a connected model such that no superset is a connected model. To obtain a γ that fulfills (5.72), it is sufficient to find all *maximal connected models* in each M_b. This method is only exemplified with one example, because the resulting γ will in many cases be γ_B.

Example 7.2 Consider the diagnostic model in Example 6.8. The structure of the diagnostic model is

Component	Assumption	Equation	x_1	x_2	z_1	z_2	z_3
Sensor 1	$\phi(s_1 = NF)$	e_1 :	X		X		
Comp	$\mathcal{B}$	e_2 :	X	X			
Sensor 2	$\phi(s_2 = NF)$	e_3 :		X		X	
Sensor 3	$\phi(s_3 = NF)$	e_4 :		X			X
	$\phi(s_3 = SG)$	e_5 :					X

(7.1)

The behavioral models M_{b_i} and the connected models of M_{b_i} are

b	M_{b_i}	maximal connected models
b_1	$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_3, e_4\}$
b_2	$\{e_2, e_3, e_4\}$	$\{e_2, e_3, e_4\}$
b_3	$\{e_1, e_2, e_4\}$	$\{e_1, e_2, e_4\}$
b_4	$\{e_1, e_2, e_3, e_5\}$	$\{e_1, e_2, e_3\}, \{e_5\}$
b_5	$\{e_2, e_4\}$	$\{e_2, e_4\}$
b_6	$\{e_2, e_3, e_5\}$	$\{e_2, e_3\}$
b_7	$\{e_1, e_2, e_5\}$	$\{e_1, e_2\}, \{e_5\}$
b_8	$\{e_2, e_5\}$	$\{e_2\}, \{e_5\}$

(7.2)

The resulting γ is the union of all models in the rightmost column in (7.2), i.e.

$$\begin{aligned}\gamma = \quad & \{\{e_1, e_2, e_3, e_4\}, \{e_2, e_3, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_3\}, \\ & , \{e_5\}, \{e_2, e_4\}, \{e_2, e_3\}, \{e_1, e_2\}, \{e_2\}\}\end{aligned} \tag{7.3}$$

When tests are designed to each of these models, it turns out that the only rejectable models are

$$\gamma = \{\{e_1, e_2, e_3, e_4\}, \{e_2, e_3, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_3\}, \{e_5\}, \{e_1, e_2\}\} \tag{7.4}$$

This γ corresponds to a sound and complete diagnostic system.

In Example 7.2 it can be seen that this method does neither propose small models nor few models. To be able to not find all maximal connected models, a more restrictive structural property for minimal rejectable models is needed.

7.2.2 Models with Spanning Tree

A common assumption when using structural analysis is that models can only be rejectable when all unknown variables can be eliminated. We need not do this assumption that generally is not true. However models where all unknown variables can be eliminated are an important type of minimal rejectable models. For connected models such that all unknown variables can be eliminated, the structure of the model contains a special type of spanning tree defined next.

Definition 7.2 (1-2-Spanning Tree Model). *If M is a model and there exists a spanning tree for $\mathcal{G}(M, \mathrm{var}_{X_u} M)$ where for each variable vertex the vertex degree is 1 or 2, the model is a **1-2-spanning tree model**.*

We will also use *2-spanning tree model* that has a spanning tree where for each variable the vertex degree is 2. For the special case when no system of equations included in a model has to be solved, a 2-spanning tree defines a substitution scheme as the next example will illustrate. However note that Definition 7.2 can be applied to models with equation systems that have to be solved.

Example 7.3 Consider the diagnostic model in Example 6.8 and especially the models $M_1 = \{e_1, e_2, e_3, e_4\}$ and $M_2 = \{e_1, e_2, e_3\}$. The bipartite graphs

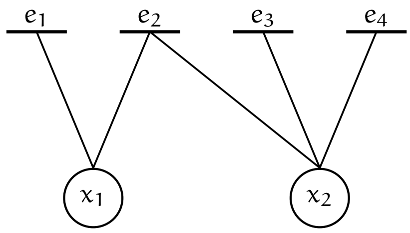

Figure 7.2 *The bipartite graph* $\mathcal{G}(M_1, \mathrm{var}_{X_u} M_1)$.

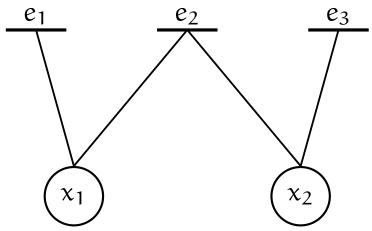

Figure 7.3 *The bipartite graph* $\mathcal{G}(M_2, \mathrm{var}_{X_u} M_2)$.

$\mathcal{G}(M_1, \mathrm{var}_{X_u} M_1)$ and $\mathcal{G}(M_2, \mathrm{var}_{X_u} M_2)$ are shown in Figure 7.2 and Figure 7.3 respectively.

In Figure 7.2, the graph $\mathcal{G}(M_1, \mathrm{var}_{X_u} M_1)$ has no spanning tree where the unknown variables have a vertex degree of 1 and 2. If all equation vertices are included in a tree, the vertex x_2 has to have a degree of 3. Hence M_1 has no 1-2-spanning tree.

The corresponding bipartite graph $\mathcal{G}(M_2, \mathrm{var}_{X_u} M_2)$ of M_2 is, as seen in Figure 7.3, a 2-spanning tree. Note that all unknown variables have a degree of 2 and no systems of equations that has to be solved simultaneously are included in this tree. Then if e_1 is considered to be the root of the tree then the following substitution scheme is define by the 2-spanning tree. Calculate x_2 using the equation e_3. Using the calculated value of x_2, e_2 can be used to calculate a value of x_1. The calculated value of x_1 is substituted into equation e_1 and model validity can be evaluated.

A model that is connected does not to have a 1-2-spanning tree, e.g. M_1 in Example 7.3, but if a model has a 1-2-spanning tree then the model is connected. Hence a more restrictive structural property than the connected property has been defined. The next question to answer is if it is true that all minimal rejectable models have a 1-2-spanning tree? Below, an example shows that there are minimal rejectable models that do not have a 1-2-spanning tree. First let the set of solutions to the model M be denoted sol$(M, \mathbf{z})$ and defined as

$$\text{sol}(M, \mathbf{z}) := \{\mathbf{x} | M(\mathbf{x}, \mathbf{z})\} \tag{7.5}$$

Example 7.4 Consider the diagnostic model

component	assumption	equation
Sensor 1	$\phi(s_1 = NF)$	$e_1 : \quad z_1 = (x+1)^2 - 1$
Sensor 2	$\phi(s_2 = NF)$	$e_2 : \quad z_2 = (x-1)^2 - 1$
Sensor 3	$\phi(s_3 = NF)$	$e_3 : \quad z_3 = x^2 - 4$

(7.6)

where the system behavioral modes are defined by

component	behavioral modes
Sensor 1	$s_1 \in \{NF, UF\}$
Sensor 2	$s_2 \in \{NF, UF\}$
Sensor 3	$s_3 \in \{NF, UF\}$

(7.7)

It can be realized that the only model that is not a 1-2-spanning tree model is $\{e_1, e_2, e_3\}$. If $\{e_1, e_2, e_3\}$ is a minimal rejectable model then an counterexample for the statement that all minimal rejectable models have a 1-2-spanning tree has been found. The model $\{e_1, e_2, e_3\}$ is a minimal rejectable model according to Definition 5.4 if there exists a $\mathbf{z}$ such that

$$\mathbf{z} \in \mathcal{O}_{\{e_1,e_2\}} \cap \mathcal{O}_{\{e_1,e_3\}} \cap \mathcal{O}_{\{e_2,e_3\}} \wedge \mathbf{z} \notin \mathcal{O}_{\{e_1,e_2,e_3\}} \tag{7.8}$$

i.e. there is a $\mathbf{z}$ such that $\{e_1, e_2, e_3\}$ is rejectable at $\mathbf{z}$ but no subset is rejectable at $\mathbf{z}$. If $\mathbf{z} = (0, 0, 0)$ then

$$\begin{aligned}
&\text{sol}(\{e_1, e_2\}, (0,0,0)) = \{0\} \\
&\text{sol}(\{e_1, e_3\}, (0,0,0)) = \{-2\} \\
&\text{sol}(\{e_2, e_3\}, (0,0,0)) = \{2\} \\
&\text{sol}(\{e_1, e_2, e_3\}, (0,0,0)) = \{-2\} \cap \{0\} \cap \{2\} = \emptyset
\end{aligned} \tag{7.9}$$

Hence $\{e_1, e_2, e_3\}$ is a minimal rejectable model but not a 1-2-spanning tree model. It can be realized that a sufficient condition for minimal rejectable models with one unknown variable to have a 1-2-spanning tree is that for each equation the solution set is connected for all observations. In this example when a solution exists the solution set is almost always not connected for all equations.

From Example 7.4, it follows that there are minimal rejectable models that are not 1-2-spanning tree models. However, there are still many models that have this property and if this structural property holds the structural advantage is, as we will see in the next section, large. Therefore it is motivated to study those models that fulfill the following assumption.

Assumption 7.1. *Let M be a model such that for any minimal rejectable model $M' \subseteq M$ it follows that M' is a 1-2-spanning tree model.*

If $\mathbb{M}$ is a diagnostic model and Assumption 7.1 holds for all behavioral models M_b, then $\mathbb{M}$ is said to fulfill Assumption 7.1. Section 7.3.2 is only devoted to linear static systems, it will be shown that such models fulfill Assumption 7.1. Next it will be shown that Assumption 7.1 is also fulfilled for a nonlinear example.

Example 7.5 The set of minimal rejectable models for the diagnostic model in Example 5.2 is

$$\gamma_m = \{\{e_5\}, \{e_3, e_4\}, \{e_1, e_2\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\} \tag{7.10}$$

as shown in (5.46). Using the structure of the diagnostic model as shown for example in (7.1) it is easy to conclude that for each $M \in \gamma_m$ it follows that M has a 1-2-spanning tree. Hence Assumption 7.1 holds for this diagnostic model.

7.3 Strategies to Find a Sound and Complete System

In the previous section it was explained that Assumption 7.1 is an important analytical property of a diagnostic model. In Section 7.3.1 an algorithm is presented that takes as input a diagnostic model that fulfills Assumption 7.1. In Section 7.3.3, methods are presented that handle diagnostic models for which it is not possible to show that Assumption 7.1 holds or not.

7.3.1 Spanning Tree Assumption Holds

In this section we consider models that fulfill Assumption 7.1. Remember that γ_m denotes the set of minimal rejectable models of $\mathbb{M}$. If a γ fulfills

$$\forall b \in \mathcal{B}(\forall M_m \in \gamma_m(M_m \subseteq M_b \rightarrow (\exists M \in \gamma : M_m \subseteq M \subseteq M_b))) \tag{7.11}$$

then γ also fulfills the conditions of Theorem 5.6. Since Assumption 7.1 is assumed to hold, all minimal rejectable models are 1-2-spanning tree models. If Σ_T denotes the set of all 1-2-spanning tree models, it follows from (7.11) that a set γ that fulfills

$$\forall b \in \mathcal{B}(\forall M_m \in \Sigma_T(M_m \subseteq M_b \rightarrow (\exists M \in \gamma : M_m \subseteq M \subseteq M_b))) \tag{7.12}$$

fulfills Theorem 5.6. From (7.12) it follows that a γ, where each model $M \in \gamma$ is maximal 1-2-spanning tree model in some behavioral model, fulfills Theorem 5.6. It is therefore sufficient to find all maximal 1-2-spanning tree models in each behavioral model.

Spanning Trees and MSS Sets

MSS sets are as described in Chapter 3 the minimal models that are structurally overdetermined. If a structurally overdetermined model is also analytical overdetermined, there is redundancy that can be used to detect inconsistencies. The correspondence between the structural and analytical property that we will use is now stated in Assumption 7.1. It is interesting to investigate the analytical properties of the MSS sets inferred by Assumption 7.1. The next theorem explains the connection between models with 1-2-spanning trees and MSS sets in structurally overdetermined models.

Theorem 7.2. *If* H *is a structurally overdetermined model such that* $|\mathrm{var}_{\mathsf{X}_u}\mathsf{H}| < \infty$*,* $\mathsf{E} \subseteq \mathsf{H}$ *is a 1-2-spanning tree model, then it follows that there exists an* $\mathsf{M} \in \mathrm{mss}\mathsf{H}$ *such that* $\mathsf{E} \subseteq \mathsf{M}$*.*

An example will illustrate Theorem 7.2, before we prove it.

Example 7.6 Consider $\mathsf{M}_{\mathbf{NF}} = \{e_1, e_2, e_3, e_4\}$ of the diagnostic model in Example 7.2. The behavioral model $\mathsf{M}_{\mathbf{NF}}$ is structurally overdetermined. All 1-2-spanning tree models are

$$
\begin{array}{llll}
\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\} & \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_4}\} & & \\
\{e_1, e_2\} & \{e_2, e_3\} & \{e_2, e_4\} & \{\mathbf{e_3}, \mathbf{e_4}\} \\
\{e_1\} & \{e_2\} & \{e_3\} & \{e_4\}
\end{array}
$$

The models that are written with bold letters are the MSS sets. Note that $\{e_2, e_4\}$ is not a maximal 1-2-spanning tree model, because $\{e_1, e_2, e_4\}$ is a 1-2-spanning tree model. It is clear that the MSS sets are exactly the maximal models that have a 1-2-spanning tree.

Note that in Example 7.6 all MSS sets have a 2-spanning tree. In Section 8.5 it will be proven that a model is an MSS sets if and only if it is a 2-spanning tree model.

The proof of Theorem 7.2 is divided into several lemmas. The idea is to build an MSS set of E by adding appropriate equations of $\mathsf{H}\backslash\mathsf{E}$. This is done by first finding a set $\mathsf{X} \subseteq \mathrm{var}_{\mathsf{X}_u}\mathsf{E}$ such that E is MSS with respect to X. If not all unknown variables of E are included in X, one of these variables x and one equation e of H is added to E such that $\mathsf{E} \cup \{e\}$ is MSS with respect to $\mathsf{X} \cup \{x\}$. Since it is assumed that there is only a finite number of unknown variables in H and therefore also of the extended set E, the number of times an unknown variable is added is finite. Hence the method will end up with an extended set $\mathsf{E} \subseteq \mathsf{H}$ that is MSS.

In Lemma 7.3 it is proven that E is MSS with respect to the variables that have degree 2 in a 1-2-spanning tree of E. In Lemma 7.4 it is described and proven how E and X can be extended with one equation e and one equation x respectively, such that $\mathsf{E} \cup \{e\}$ is MSS with respect to $\mathsf{X} \cup \{x\}$. In Lemma 7.5 it is proven that the result of the extending method described in Lemma 7.4 is either an MSS set or fulfills the conditions needed to apply the method in Lemma 7.4 once more.

Lemma 7.3. *Let* E *be a model that has a 1-2-spanning tree* T *and let* X *be the set of variables that has degree 2 in* T*, then it follows that* E *is MSS with respect to* X*.*

Proof. According to Theorem 3.4 it follows that E is MSS with respect to X if and only if for all $e \in \mathsf{E}$ there exists a perfect matching in $\mathcal{G}(\mathsf{E}\backslash\{e\}, \mathsf{X})$.

Now we will prove that there is a complete matching of X into E. Using Corollary 3.3 it is equivalent to show that for any set $X' \subseteq X$ it holds that $|X'| \leq |\text{equ}_E(X')|$. Let X' be an arbitrary subset of X. The *subgraph* (Grimaldi 1994) of T induced by the vertices $\text{equ}_E(X') \cup X'$ is a *forest* (Grimaldi 1994) consisting of the trees $T_1, \ldots, T_n$. Let X_i be the set of variables vertices included in tree T_i.

For any tree it holds that the number of vertices is equal to the number of edges plus 1. Using the fact that all variable vertices have degree 2, it follows that the number of edges in T_i is $2\,|X_i|$. The number of vertices are $|\text{equ}_{T_i}(X_i)| + |X_i|$ which, gives using the equation describing how the number of edges depends on the number of vertices in a tree, that

$$|\text{equ}_{T_i}(X_i)| + |X_i| = 2\,|X_i| + 1 \Leftrightarrow |\text{equ}_{T_i}(X_i)| = |X_i| + 1 \tag{7.13}$$

Since (7.13) is valid for each tree in the forest it follows that $|X'| + n = |\text{equ}_T(X')|$. Since the number of trees n is at least 1 it follows that $|X'| \leq |\text{equ}_T(X')|$ holds. Hence there is a complete matching of X into E in T.

Take any complete matching from X into E in T. We will show that this complete matching can be used to construct a perfect matching in $\mathcal{G}(E\backslash\{e\}, X,)$. Hence take any $e \in E$. If e is not included in the complete matching then the complete matching is a perfect matching in $\mathcal{G}(E\backslash\{e\}, X,)$. Assume therefore that e is included in the complete matching. By first removing all variable vertices with degree 1 in T and then e, T' is obtained. Assume that x was assigned to e in T. Let the connected component of T' that contains x be denoted T''.

Note that T'' by the construction is a tree. The vertex x has degree 1 in T''. The variable x is the only variable in T'' with degree 1 and it can be realized as follows. If two variables have degree 1 in T'' it means that both variables are connected to e in T. Hence there is two paths between these two variables, one that is contained in T'' and one that goes through e. Both these paths are contained in T which then contains a cycle. A contradiction is derived because T is assumed to be a tree. It follows that x is the only variable vertex of degree 1 in T''.

Since T'' is a subgraph of T and all other variable vertices in T have a degree of 2 it follows that there exists an augmented path between x and an unmatched equation. Switching the matched and non-matched edges in the augmented path will produce a perfect matching in the subgraph of T induced by the vertices $E\backslash\{e\} \cup X$. Since e was arbitrarily chosen it follows that for any $e \in E$ there is a perfect matching in $\mathcal{G}(E\backslash\{e\}, X)$. Hence E is MSS with respect to X. □

Lemma 7.4. *Let H be a structurally overdetermined set. If $E \subseteq H$ is a set of equations and $X \subset \text{var}_{X_u} E$ is a set of variables such that E is MSS with respect to X, then it follows that there is an $x \in \text{var}_{X_u}(E)\backslash X$ and an $e \in H\backslash E$ such that $E \cup \{e\}$ is MSS with respect to $X \cup \{x\}$.*

Proof. According to (3.5), the statement that the set H is structurally overdetermined can formally be written as

$$\forall X' \subseteq \text{var}_{X_u}(H), X' \neq \varnothing : |X'| < |\text{equ}_H(X')|, \tag{7.14}$$

Since E is MSS with respect to the proper subset X of unknown variables in E, i.e. $\text{var}_{X_u} E$, it follows from Theorem 3.4 that there is a complete matching of E into $\text{var}_{X_u} E$. From Corollary 3.2 it follows that

$$\forall E' \subseteq E : |E'| \leq |\text{var}_{X_u}(E')|, \tag{7.15}$$

holds. The fact that E is MSS with respect to X can be stated in the following equivalent way. Considering only the variables X as unknown variables, E is structurally overdetermined, i.e.

$$\forall X' \subseteq X, |X'| \neq 0 : |X'| < |\text{equ}_E(X')|. \tag{7.16}$$

and

$$|E| = |X| + 1 \tag{7.17}$$

To show that $E \cup \{e\}$ is MSS with respect to $X \cup \{x\}$ can therefore be shown by proving that

$$\forall X' \subseteq X \cup \{x\}, |X'| \neq 0 : |X'| < |\text{equ}_{E \cup \{e\}}(X')| \tag{7.18}$$

and

$$|E \cup \{e\}| = |X \cup \{x\}| + 1. \tag{7.19}$$

holds.

First we will find an x such that $x \notin X$ and $x \in \text{var}_{X_u} E$. Using (7.17) and $|E| \leq |\text{var}_{X_u} E|$ derived from (7.15), it is clear that $|X| < |E| \leq |\text{var}_{X_u} E|$. Hence there must be an $x \in \text{var}_{X_u}(E) \backslash X$. Take an arbitrary $x \in \text{var}_{X_u}(E) \backslash X$.

Let a set of variables X', where $X' \neq \varnothing$ and $X' \subseteq X \cup \{x\}$, be called a *critical set* if

$$|\text{equ}_E(X')| = |X'|. \tag{7.20}$$

There exists always a critical set because $X' = X \cup \{x\}$ is a critical set.

Next, we will show that there is a unique *minimal critical set*. Suppose there are two minimal critical sets X_{c1} and X_{c2} where $X_{c1} \neq X_{c2}$. Note that $X_{ci} \neq \varnothing$ for $i \in \{1, 2\}$ to satisfy (7.20).

Suppose that $X' \subset X \cup \{x\}$ is a critical set such that $x \notin X'$. Then $X' \subseteq X$ and (7.16) can be used deriving $|X'| < |\text{equ}_E(X')|$. This contradicts the fact that X' is critical. Hence all critical sets include x.

Then it is possible to do the following partition of $X_{c1} \cup X_{c2}$, denoting $X_{c1} \cap X_{c2} = X_{12} \cup \{x\}$ where $x \notin X_{12}$, $X_1 = X_{c1} \backslash X_{c2}$, and $X_2 = X_{c2} \backslash X_{c1}$. Figure 7.4 visualizes the partition.

According to the partition, the critical sets X_{c1} and X_{c2} are expressed as

$$X_{c1} = X_1 \cup X_{12} \cup \{x\} \tag{7.21}$$

and

$$X_{c2} = X_2 \cup X_{12} \cup \{x\}. \tag{7.22}$$

From (7.20), (7.21), (7.22), and the fact that X_{ci} is critical it follows that

$$|\text{equ}_E(X_{ci})| = |X_{ci}| = |X_i| + |X_{12}| + 1 \quad \text{for} \quad i \in \{1, 2\}. \tag{7.23}$$

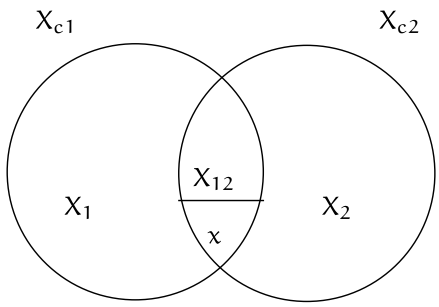

Figure 7.4 *The partition of* X_{c1} *and* X_{c2}*.*

Since X_{c2} is a minimal critical set it follows that $X_{c1} \not\subseteq X_{c2}$. Using (7.21) and (7.22) and the knowledge that these are partitions, imply the equivalent expression $X_1 \neq \varnothing$. This implies that

$$X_{12} \cup \{x\} \subset X_1 \cup X_{12} \cup \{x\} = X_{c1}. \tag{7.24}$$

Consider first any subset $X' \neq \varnothing$ of $X \cup \{x\}$ such that $X' \neq \{x\}$. From (7.16) it follows that

$$|X'| \leq |X' \backslash \{x\}| + 1 \leq |\mathrm{equ}_E(X' \backslash \{x\})| \leq |\mathrm{equ}_E(X')|. \tag{7.25}$$

Further on, if $X' = \{x\}$ then

$$|\{x\}| \leq |\mathrm{equ}_E(\{x\})|, \tag{7.26}$$

because of the fact that x is chosen such that $x \in \mathrm{var}_{X_u} E$. The inequalities (7.25) and (7.26) implies that

$$\forall X' \subseteq X \cup \{x\}, X' \neq \varnothing : |X'| \leq |\mathrm{equ}_E(X')|. \tag{7.27}$$

Now, the minimality of X_{c1} and (7.24) imply that $X_{12} \cup \{x\}$ is not critical, i.e.

$$|\mathrm{equ}_E(X_{12} \cup \{x\})| \neq |X_{12} \cup \{x\}|. \tag{7.28}$$

The set $X_{12} \cup \{x\}$ satisfies (7.27) and (7.28), hence

$$|\mathrm{equ}_E(X_{12} \cup \{x\})| \geq |X_{12} \cup \{x\}| + 1 = |X_{12}| + 2. \tag{7.29}$$

From the definition of the function equ it follows that for arbitrary variable sets A and B and for an arbitrary equation set $\bar{E}$ it holds that

$$\mathrm{equ}_{\bar{E}}(A \cup B) = \mathrm{equ}_{\bar{E}}(A) \cup \mathrm{equ}_{\bar{E}}(B). \tag{7.30}$$

Using (7.30) and basic set theory imply that

$$\begin{gathered} |\mathrm{equ}_E(X_1 \cup X_{12} \cup \{x\}) \cup \mathrm{equ}_E(X_2 \cup X_{12} \cup \{x\})| = \\ |\mathrm{equ}_E(X_1 \cup X_{12} \cup \{x\} \cup X_2 \cup X_{12} \cup \{x\})| = \\ |\mathrm{equ}_E(X_1 \cup X_2 \cup X_{12} \cup \{x\})|. \end{gathered} \tag{7.31}$$

Further on, it holds that

$$\begin{gathered}\mathrm{equ}_E(X_1 \cup X_{12} \cup \{x\}) \cap \mathrm{equ}_E(X_2 \cup X_{12} \cup \{x\}) = \\ \big(\mathrm{equ}_E(X_1) \cup \mathrm{equ}_E(X_{12} \cup \{x\})\big) \cap \big(\mathrm{equ}_E(X_2) \cup \mathrm{equ}_E(X_{12} \cup \{x\})\big) = \\ \big(\mathrm{equ}_E(X_1) \cap \mathrm{equ}_E(X_2)\big) \cup \mathrm{equ}_E(X_{12} \cup \{x\}).\end{gathered} \tag{7.32}$$

The last row in (7.32) can be estimated from below using (7.29)

$$\begin{gathered}\left|\big(\mathrm{equ}_E(X_1) \cap \mathrm{equ}_E(X_2)\big) \cup \mathrm{equ}_E(X_{12} \cup \{x\})\right| \geq \\ |\mathrm{equ}_E(X_{12} \cup \{x\})| \geq |X_{12}| + 2.\end{gathered} \tag{7.33}$$

Now, we will apply $|A \cup B| = |A| + |B| - |A \cap B|$, where $A = \mathrm{equ}_E(X_1 \cup X_{12} \cup \{x\})$ and $B = \mathrm{equ}_E(X_2 \cup X_{12} \cup \{x\})$. The left hand side $|A \cup B|$ can be simplified using (7.31). The result is

$$\begin{gathered}|\mathrm{equ}_E(X_1 \cup X_2 \cup X_{12} \cup \{x\})| = |\mathrm{equ}_E(X_1 \cup X_{12} \cup \{x\})| + \\ |\mathrm{equ}_E(X_2 \cup X_{12} \cup \{x\})| - \\ |\mathrm{equ}_E(X_1 \cup X_{12} \cup \{x\}) \cap \mathrm{equ}_E(X_2 \cup X_{12} \cup \{x\})|.\end{gathered} \tag{7.34}$$

Further, substitute the results in (7.21), (7.22), (7.23), and (7.32) into (7.34), then

$$\begin{gathered}|\mathrm{equ}_E(X_1 \cup X_2 \cup X_{12} \cup \{x\})| = |X_1| + |X_{12}| + 1 + |X_2| + \\ |X_{12}| + 1 - |\big(\mathrm{equ}_E(X_1) \cap \mathrm{equ}_E(X_2)\big) \cup \mathrm{equ}_E(X_{12} \cup \{x\})|.\end{gathered} \tag{7.35}$$

The last part of (7.35) is estimated from above using (7.33)

$$\begin{gathered}|X_1| + |X_2| + 2|X_{12}| + 2 - \\ |\big(\mathrm{equ}_E(X_1) \cap \mathrm{equ}_E(X_2)\big) \cup \mathrm{equ}_E(X_{12} \cup \{x\})| \\ \leq |X_1| + |X_2| + 2|X_{12}| + 2 - (|X_{12}| + 2) = |X_1| + |X_2| + |X_{12}|.\end{gathered} \tag{7.36}$$

The result of putting (7.35) and (7.36) together is

$$\begin{gathered}|X_1| + |X_2| + |X_{12}| \geq |\mathrm{equ}_E(X_1 \cup X_2 \cup X_{12} \cup \{x\})| \\ \geq |\mathrm{equ}_E(X_1 \cup X_2 \cup X_{12})|.\end{gathered} \tag{7.37}$$

Finally, $X_1 \cup X_2 \cup X_{12} \subseteq X$ and according to (7.16) is

$$|X_1| + |X_2| + |X_{12}| < |\mathrm{equ}_E(X_1 \cup X_2 \cup X_{12})|. \tag{7.38}$$

The inequalities (7.37) and (7.38) implies a contradiction. Hence there cannot be two minimal critical sets. Let the unique minimal critical set be denoted $X_{critical}$.

Now, it is time to show that there exists an equation $e \in H \backslash E$ that fulfill (7.18). Suppose that $\mathrm{equ}_H(X_{critical}) \subseteq E$. This together with (7.20) implies that $|X_{critical}| = |\mathrm{equ}_E(X_{critical})| = |\mathrm{equ}_H(X_{critical})|$. This is a contradiction according to (7.14). Hence there is an equation $e \in H \backslash E$ such that $\mathrm{var}_{X_{critical}}(e) \neq \emptyset$. $X_{critical} \cap \mathrm{var}_{X \cup \{x\}}(e) \neq \emptyset$.

Now, we will show that this e fulfills (7.18). Take an arbitrary $X' \subseteq X \cup \{x\}$ where $X' \neq \varnothing$. Consider first the case when $X_{critical} \subseteq X'$. Then it follows from (7.27) that $|X'| \leq |\text{equ}_E(X')| < |\text{equ}_{E\cup\{e\}}(X')|$. The last inequality follows from the fact that $\text{equ}_{\{e\}}(X_{critical}) = \{e\}$.

The opposite case is when $X_{critical} \not\subseteq X'$. From the fact that there is a unique minimal critical set it follows that X' cannot be a critical set. Hence (7.27) and $|X'| \neq |\text{equ}_E(X')|$ conclude that $|X'| < |\text{equ}_E(X')| \leq |\text{equ}_{E\cup\{e\}}(X')|$, i.e. (7.18) holds.

Finally, it remains to prove (7.19). Simple calculations using (7.17) gives $|E \cup \{e\}| = |E|+1 = |X|+2 = |X\cup\{x\}|+1$. Hence $E\cup\{e\}$ is MSS with respect to $X\cup\{x\}$. □

Lemma 7.5. *Let H, $E_j \subseteq H$, and $X_j \subseteq \text{var}_{X_u} E_j$ fulfill the conditions in Lemma 7.4. If $E_{j+1} = E_j \cup \{e\}$ and $X_{j+1} = X_j \cup \{x\}$ where x and e are defined in Lemma 7.4, then E_{j+1} is either an MSS set, with respect to the unknown variables, or fulfills the conditions in Lemma 7.4.*

Proof. Note that according to Lemma 7.4, (7.19) and (7.18) hold for the set E_{j+1} and X_{j+1}. Take any $\hat{E} \subset E_{j+1}$. Then there is an $e \in E_{j+1}\backslash\hat{E}$ such that $\hat{E} \subseteq E_{j+1}\backslash\{e\}$. From (7.18) it follows that

$$\forall X' \subseteq X_{j+1}, X' \neq \varnothing : |X'| \leq |\text{equ}_{E_{j+1}}(X')| - 1 \leq |\text{equ}_{E_{j+1}\backslash\{e\}}(X')| \tag{7.39}$$

Especially, if $X' = X_{j+1}$ in (7.39) then

$$|X_{j+1}| \leq |\text{equ}_{E_{j+1}\backslash\{e\}}(X_{j+1})| \tag{7.40}$$

holds. From (7.19) it follows that

$$|\text{equ}_{E_{j+1}\backslash\{e\}}(X_{j+1})| \leq |E_{j+1}\backslash\{e\}| = |X_{j+1}|. \tag{7.41}$$

The inequalities (7.40) and (7.41) imply

$$|X_{j+1}| = |\text{equ}_{E_{j+1}\backslash\{e\}}(X_{j+1})|. \tag{7.42}$$

Now, using (7.39) in Corollary 3.3 it follows that there is a complete matching of X_{j+1} into $E_{j+1}\backslash\{e\}$. The complete matching is also a perfect matching, according to (7.42). A perfect matching is especially a complete matching of $E_{j+1}\backslash\{e\}$ into X_{j+1}. Corollary 3.2 implies that

$$\forall E' \subseteq E_{j+1}\backslash\{e\} : |E'| \leq |\text{var}_{X_{j+1}}(E')|. \tag{7.43}$$

Since $\hat{E} \subseteq E_{j+1}\backslash\{e\}$, then $E' = \hat{E}$ in (7.43) implies that

$$|\hat{E}| \leq |\text{var}_{X_{j+1}}(\hat{E})|. \tag{7.44}$$

The set $\hat{E}$ was an arbitrary proper subset to E_{j+1}. This implies that

$$\forall E' \subset E_{j+1} : |E'| \leq |\text{var}_{X_{j+1}}(E')| \leq |\text{var}_{X_u}(E')|. \tag{7.45}$$

Now, it remains to study $|\mathrm{var}_{X_u}(E_{j+1})|$. From (7.45) it holds that

$$|E_{j+1}| = |E_j| + 1 \leq |\mathrm{var}_{X_{j+1}}(E_j)| + 1 \leq |\mathrm{var}_{X_{j+1}}(E_{j+1})| + 1 \leq |\mathrm{var}_{X_u}(E_{j+1})| + 1. \tag{7.46}$$

There are two cases. Suppose that equality in (7.46) holds, i.e.

$$|E_{j+1}| = |\mathrm{var}_{X_u}(E_{j+1})| + 1. \tag{7.47}$$

From (7.45), (7.47), and the definition of MSS sets it follows that E_{j+1} is an MSS set.

Next assume that, (7.46) is a strict inequality, i.e.

$$|E_{j+1}| \leq |\mathrm{var}_{X_u}(E_{j+1})|. \tag{7.48}$$

Hence according to (7.45) and (7.48) it follows that

$$\forall E' \subseteq E_{j+1} : |E'| \leq |\mathrm{var}_{X_u}(E')|. \tag{7.49}$$

□

Now it is time to prove Theorem 7.2.

Proof. Let T be a 1-2-spanning tree for E, such that Definition 7.2 is satisfied. According to Lemma 7.3 there is a set $X \subseteq \mathrm{var}_{X_u} E$ such that E is MSS with respect to X. If $X = \mathrm{var}_{X_u} E$, E is an MSS set and there is nothing more to prove. Assume therefore that $X \subset \mathrm{var}_{X_u} E$. Then H, E, and X fulfill the conditions in Lemma 7.4. This means that if $E_1 = E$ and $X_1 = X$, Lemma 7.5 can repeatedly be applied until there is for some $i \geq 2$ an MSS set E_i such that $E \subset E_i \subseteq H$. That is, for some i, E_i will be MSS with respect to X_u. The existence of such an i can be proven as follows. Each time the method that Lemma 7.5 describes is applied, one additional unknown variable will be included. Since we have assumed that $|\mathrm{var}_{X_u} H| < \infty$ it follows that $\mathrm{var}_{X_u} H$ is an upper bound for i. □

The conclusion of Theorem 7.2 is that in a structurally overdetermined model the maximal models that has a 1-2-spanning tree are subsets to MSS sets. If $\mathbb{M}$ is a diagnostic model, $M \subseteq^* \mathbb{M}$ then $\mathrm{mss}M$ denotes the MSS sets that correspond to a feasible models. Moreover $\mathrm{mss}\mathbb{M} := \cup_{b \in \mathcal{B}} \mathrm{mss}M_b$.

Corollary 7.6. *If $\mathbb{M}$ is a diagnostic model that fulfills Assumption 7.1 and where $M_b^* \subseteq M_b^+$ for all system behavioral modes $b \in \mathcal{B}$, then $\gamma = \mathrm{mss}\mathbb{M}$ gives a sound and complete diagnostic system.*

Proof. From the definition of the set M_b^* in (6.33) it follows that M_b^* is a superset of all detection models for b that are needed to find all inconsistencies of M_b. From the definition of detection model it follows that detection models are minimal rejectable models. From Assumption 7.1 it follows that all minimal rejectable models are tree models. Hence all detection models are tree models. From condition

$M_b^* \subseteq M_b^+$ and Theorem 7.2 it follows that all detection models for b are subsets of MSS sets. Since this holds for all system behavioral modes it follows that $\gamma = \text{mss}\mathbb{M}$ fulfills (5.72) and it follows from Theorem 5.6 that γ corresponds to a sound and complete diagnostic system. □

This corollary gives the results needed to prove that the algorithm presented next finds a γ that corresponds to a sound and complete system.

Algorithm 7.1.
Input: $\mathbb{M}$ *that fulfills Assumption 7.1.*

a) *Set* $\gamma := \text{mss}\mathbb{M}$.

b) *For each behavioral mode* $b \in \mathcal{B}$ *if (6.109) in Theorem 6.12 is false when* $\hat{M} = M_b^+$*, then add to* γ *all the maximal models that has a 1-2-spanning tree and that has a non-empty intersection with* $M_b \backslash M_b^+$.

Output: γ

To check that the condition $M_b^* \subseteq M_b^+$ in Corollary 7.6 is fulfilled, Theorem 6.12 is used by validating (6.109) when $\hat{M} = M_b^+$. Note that (6.109) in step (b) is an analytical condition. Even if the (6.109) is the condition in Theorem 6.12 that relies on Assumption 6.1, it can be realized that Assumption 6.1 need not be fulfilled to find a set γ that corresponds to a complete and sound diagnostic system using Algorithm 7.1. If $M_b^* \subseteq M_b^+$ holds it follows that the MSS sets contained in M_b is sufficient according to Corollary 7.6. If $M_b^* \subseteq M_b^+$ cannot be validated, i.e. the condition of Corollary 7.6 cannot be validated, the goal is to find all maximal 1-2-spanning tree models according to the conclusion in Section 7.3.1. Step (b) is designed to add all maximal 1-2-spanning tree models to γ.

Example 7.7 Consider the diagnostic model $\mathbb{M}$ in Example 7.2. From Example 7.5 it follows that Assumption 7.1 is fulfilled. Then Algorithm 7.1 can be applied to $\mathbb{M}$. In step (a) the resulting MSS sets are

$$\text{mss}\mathbb{M} = \{\{e_5\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\} \tag{7.50}$$

In Example 6.8, the test in step (b) in Algorithm 7.1 is performed for each behavioral mode. All passed except the test for behavioral mode b_7. The maximal tree model that has a nonempty intersection with $M_{b_7} \backslash M_{b_7}^+$ is $\{e_1, e_2\}$. This set together with the MSS sets is the output of Algorithm 7.1, i.e.

$$\gamma = \{\{e_5\}, \{e_1, e_2\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\} \tag{7.51}$$

Note that this set γ is equal to γ_m. To compute γ the only analytical calculations done are the calculations done in Example 6.8.

The next section describes how Algorithm 7.1 can be simplified for the special case when the equations of the diagnostic model are linear static equations.

7.3.2 Linear Static Example

To be able to apply Algorithm 7.1 to a diagnostic model, we first have to prove that the diagnostic model fulfills Assumption 7.1. The next theorem implies that diagnostic models that have linear static equations fulfill Assumption 7.1.

Theorem 7.7. *If a minimal rejectable model* M *can be written on the form*

$$M : A\,\mathbf{x} + B\,\mathbf{z} + \mathbf{c} = 0 \tag{7.52}$$

where A *and* B *are constant matrices and* $\mathbf{c}$ *is a constant vector, then it follows that* M *has a 1-2-spanning tree, i.e.* M *fulfills Assumption 7.1.*

Proof. A model M is minimal rejectable if and only if there exists a $\mathbf{z} = \mathbf{z}_0$ such that

$$\forall \mathbf{x} \neg M(\mathbf{x}, \mathbf{z}_0) \tag{7.53}$$

and for all $M_i = M \backslash \{e_i\}$

$$\exists \mathbf{x} M_i(\mathbf{x}, \mathbf{z}_0) \tag{7.54}$$

If $(B\,\mathbf{z}_0 + \mathbf{c})$ is denoted $\mathbf{b}$ then expression (7.53) can be rewritten as

$$\text{rank}(A) < \text{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) \tag{7.55}$$

If D is a matrix or vector, e.g. A, B, or $\mathbf{c}$, M is a set of equations corresponding to rows in D, X a set of variables corresponding to columns in D, then $D(M, X)$ is the matrix consisting of the rows corresponding to M and the columns corresponding to X in the matrix D. Further let $D(M)$ be the matrix consisting of all rows of D corresponding to M and let $D(X)$ be the matrix consisting of all columns of D corresponding to X.

With the new notation, expression (7.54) can be rewritten as

$$\text{rank}(A(M_i)) = \text{rank}(\begin{bmatrix} A(M_i) & \mathbf{b}(M_i) \end{bmatrix}) \tag{7.56}$$

Elementary linear algebra implies the inequalities

$$\text{rank}(A(M_i)) \leq \text{rank}(A) \tag{7.57}$$

and

$$\text{rank}(\begin{bmatrix} A(M_i) & \mathbf{b}(M_i) \end{bmatrix}) \leq \text{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) \leq \text{rank}(\begin{bmatrix} A(M_i) & \mathbf{b}(M_i) \end{bmatrix}) + 1 \tag{7.58}$$

From (7.58) it is clear that

$$\text{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) = \begin{cases} \text{rank}([A(M_i) \quad \mathbf{b}(M_i)]) & \text{or} \\ \text{rank}([A(M_i) \quad \mathbf{b}(M_i)]) + 1 & \end{cases} \tag{7.59}$$

Assume that

$$\text{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) = \text{rank}(\begin{bmatrix} A(M_i) & \mathbf{b}(M_i) \end{bmatrix}) \tag{7.60}$$

holds. From (7.56) and (7.60) it follows that

$$\operatorname{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) = \operatorname{rank}(A(M_i)) \tag{7.61}$$

From (7.57) and (7.61) it follows that

$$\operatorname{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) \leq \operatorname{rank}(A) \tag{7.62}$$

Finally from (7.55) and (7.62) it follows that

$$\operatorname{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) < \operatorname{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) \tag{7.63}$$

which is a contradiction. Hence the assumption that (7.60) is false. From (7.59) it follows then that

$$\operatorname{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) = \operatorname{rank}(\begin{bmatrix} A(M_i) & \mathbf{b}(M_i) \end{bmatrix}) + 1 \tag{7.64}$$

Since (7.64) holds for any $e_i \in M$ it follows that

$$\operatorname{rank}(\begin{bmatrix} A & \mathbf{b} \end{bmatrix}) = |M| \tag{7.65}$$

We will show that it is possible to construct a spanning tree in $\mathcal{G}(M, \operatorname{var}_{X_u} M)$ with vertex degree 1 or 2 for all variable vertices. Those variable vertices that have degree 2 are the variables that connect two equations. This set of variables cannot be chosen arbitrarily. Let $X_2 \subseteq \operatorname{var}_{X_u} M$ be the variables that will have degree 2 in the tree to be constructed. From (7.55) it follows that a set X_2 can be chosen such that $\begin{bmatrix} A(X_2) & \mathbf{b} \end{bmatrix}$ is a quadratic full rank matrix, i.e.

$$\operatorname{rank}(\begin{bmatrix} A(X_2) & \mathbf{b} \end{bmatrix}) = |M| \tag{7.66}$$

There is such an X_2 because $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ has full row-rank. The matrix A and some of the defined equation sets and variable sets are shown in Figure 7.5.

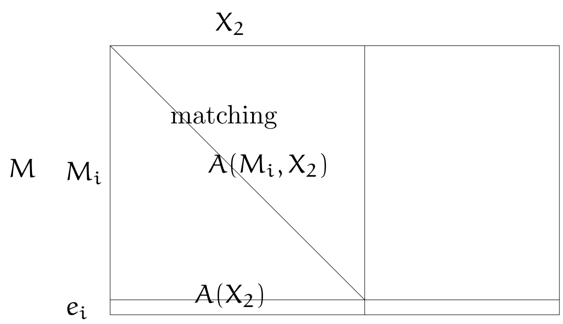

Figure 7.5 *A sketch of matrix A.*

Since $\begin{bmatrix} A(X_2) & \mathbf{b} \end{bmatrix}$ is a full rank square matrix it follows that

$$\operatorname{rank}(A(X_2)) = |M| - 1 \tag{7.67}$$

From (7.55) and (7.65) it follows that

$$\text{rank}(A) = |M| - 1 \tag{7.68}$$

Equalities (7.67) and (7.68) imply that there exists a matrix F such that

$$A(\text{var}_{X_u} M \backslash X_2) = A(X_2)\, F \tag{7.69}$$

This implies that

$$A(M_i, \text{var}_{X_u} M \backslash X_2) = A(M_i, X_2)\, F \tag{7.70}$$

and therefore it follows that

$$\text{rank}(A(M_i)) = \text{rank}(A(M_i, X_2)) \tag{7.71}$$

Equality (7.65) implies using (7.56) and (7.64) that

$$\text{rank}(A(M_i)) = |M| - 1 = |M_i| \tag{7.72}$$

Then (7.71) and (7.72) imply that

$$\text{rank}(A(M_i, X_2)) = |M| - 1 = |M_i| \tag{7.73}$$

The last equality will turn out to be an important property to be able to construct a tree.

The *structural rank* for A is defined as the size of a maximum matching in $\mathcal{G}(M, \text{var}_{X_u} M)$ and will be denoted srank(A). From the definitions of rank and structural rank it follows that

$$\text{srank}(A) \geq \text{rank}(A) \tag{7.74}$$

From (7.73) and (7.74) it follows that there exists a complete matching of M_i into X_2. Assume that a tree is constructed starting with $e_1 \in M$. Then if $i = 1$ in (7.73) it follows that there is a perfect matching P in $\mathcal{G}(M_1, X_2)$. Next a constructive algorithm is presented that builds a spanning tree. To have an example to think about when reading the algorithm description, a graph is shown in Figure 7.6. The edges marked in bold are P in this example.

Given P, a spanning tree is constructed in the following way:

- Let T_0 be the a graph with no vertices and let T_1 be a tree with the vertex e_1 and no edges. The tree T_1 in the example is the equation vertex e_1.
- Given a tree T_i, the next tree T_{i+1} is defined in the following way. Let the equation vertices contained in T_i, but not in T_{i-1}, be denoted E. Add to the tree T_i, all variable vertices V that is not included in T_i and that, in $\mathcal{G}(M, \text{var}_{X_u} M)$, are adjacent to some vertex of E. To the resulting graph, add one edge included in $\mathcal{G}(M, \text{var}_{X_u} M)$ to each new variable vertex in V such that the edge connects the new variable vertex with any equation vertex in E. Finally, to obtain T_{i+1}, add all edges in P and corresponding equation vertices in P that are incident with some vertex in V.

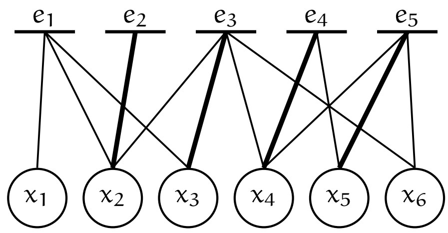

Figure 7.6 *A bipartite graph. The bold marked edges are* P *defined in .*

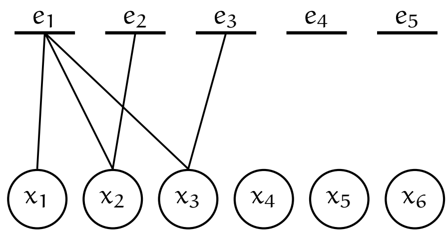

Figure 7.7 *The tree* T_2 *of the graph in Figure 7.6.*

- The tree T_{i+1} is a 1-2-spanning tree for M when all equations in M have a corresponding vertex in T_{i+1}.

The consecutive tree constructed of this algorithm on the graph in Figure 7.6 is shown in Figures 7.7-7.9. It is clear that the variable vertex degrees are 1 or 2 in all T_i. Therefore the only way the algorithm could fail to find a 1-2-spanning tree for M is if the numbers of equation vertices in T_i are equal to the equation vertices in T_{i+1} for some i such that not all equations in M are included in T_i. Now, we will prove that this case is impossible and hence the algorithm will always find a 1-2-spanning tree.

Assume that the numbers of equation vertices in T_i are equal to the equation vertices in T_{i+1} and the number of equation vertices are less than $|M|$. Denote this

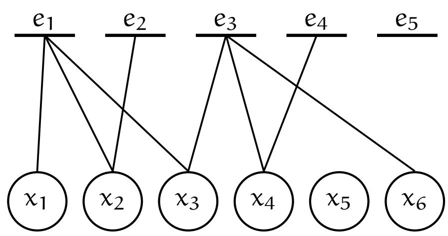

Figure 7.8 *The tree* T_3 *of the graph in Figure 7.6.*

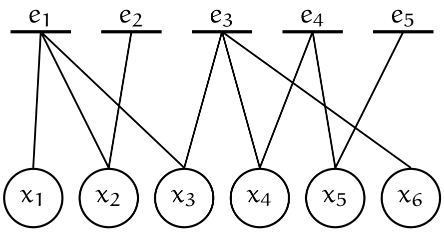

Figure 7.9 *A spanning tree T_4 of the graph in Figure 7.6 with degree 1 or 2 for all variable vertices.*

set of equations $\bar{M} \subset M$. In Figure 7.10 $\bar{M}$ among other notions are illustrated. According to the description of T_{i+1}, it follows that $\text{var}_{X_2}\bar{M}$ do not include any of the variables assigned to equations in $M \backslash \bar{M}$. But then it follows that

$$|\bar{M}| = |\text{var}_{X_2}\bar{M}| + 1 \tag{7.75}$$

In Figure 7.10 the conclusion (7.75) is illustrate with the zero. This means that

$$\text{srank}(A(\bar{M}, X_2)) < |\bar{M}| \tag{7.76}$$

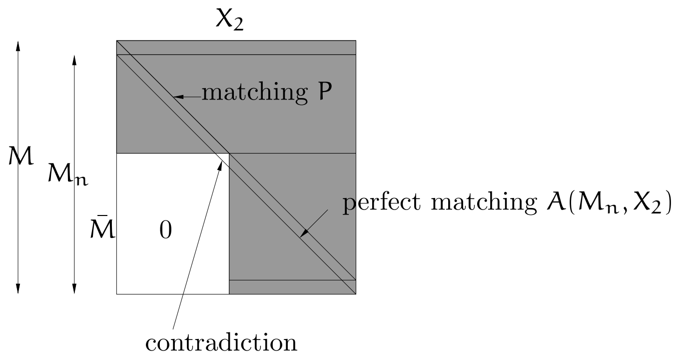

Figure 7.10 *A sketch of matrix $A(X_2)$.*

Since (7.73) holds for any M_i, there exists an M_n such that $\bar{M} \subseteq M_n$. Using M_n in (7.73) it follows that $A(M_n, X_2)$ is a full rank matrix. The matrix $A(M_n, X_2)$ includes the rows of $A(\bar{M}, X_2)$. Hence $A(\bar{M}, X_2)$ has full row rank, i.e.

$$\text{rank}(A(\bar{M}, X_2)) = |\bar{M}| \tag{7.77}$$

From (7.74) and (7.77) it follows that

$$\text{srank}(A(\bar{M}, X_2)) \geq |\bar{M}| \tag{7.78}$$

Now, (7.76) and (7.78) is a contradiction. Hence T_{i+1} will always include more equation vertices than T_i until all equations in M are included. Since M was an arbitrary minimal rejectable linear static model it is shown that all minimal rejectable linear static models M, are 1-2-spanning tree models. □

From Theorem 7.7 it follows that Algorithm 7.1 can always be applied to linear static models. Further the test (6.109) in Theorem 6.12 is in this case transformed to test if $A(\bar{M}, \bar{X})$ has full row-rank.

Example 7.8 Consider the following diagnostic model that consists of 15 static linear equations

$$\begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \\ e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \\ e_{15} \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.92 & 0 & 0.78 & 0 & 0 \\ 0.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.74 & 0.44 & 0 & 0 & 0 & 0 & 0.59 & 0.11 & 0 \\ 0.32 & 0 & 0 & 0.62 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.89 & 0.96 & 0 & 0 & 0.02 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.11 & 0 & 0.4 & 0.097 \\ 0 & 0 & 0.29 & 0.68 & 0 & 0.98 & 0.4 & 0 & 0 & 0 \\ 0 & 0.82 & 0 & 0 & 0 & 0 & 0.13 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.84 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.23 & 0.067 & 0.33 & 0.58 & 0 & 0.14 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.48 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.97 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.58 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \tag{7.79}$$

The system behavioral modes that are considered are that each equation can be faulty. This means that there are $2^{15} = 32768$ system behavioral modes. We want to design a sound and complete diagnostic system with respect to the diagnostic model given in (7.79). To make one test for each behavioral mode that can be rejected the total number of tests will in this case be 14292. Using Algorithm 7.1 the number of tests in the resulting sound and complete diagnostic system is 62. In this linear static example the analytical test used in step (b) in Algorithm 7.1 for each behavioral mode is

$$\operatorname{rank}(A(M_b \backslash M_b^+, X_u \backslash \operatorname{var}_{X_u} M_b^+)) = |M_b \backslash M_b^+| \tag{7.80}$$

This expression is equivalent to (6.115) that implies (6.109. In this example expression 7.80 was true for all system behavioral modes. The 62 acceptance sets are defined with the following 62 equations

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0.65 & 0 & 0 \\
0.29 & 0 & -0.56 & 0 & 0 & 0 & 0 \\
0.31 & 0 & -0.59 & 0 & -0.5 & 0 & 0 \\
0 & 0 & 0 & -0.1 & 0 & -0.37 & -0.089 \\
0 & 0.33 & -0.58 & -0.23 & 0 & 0 & 0 \\
0.26 & -0.28 & 0 & 0.2 & 0 & 0 & 0 \\
0.27 & -0.29 & 0 & 0.2 & -0.43 & 0 & 0 \\
0.26 & 0 & 0 & 0.15 & 0 & 0.044 & 0 \\
0.16 & 0 & 0.25 & 0.16 & 0 & 0.049 & 0 \\
0.26 & 0 & 0 & 0.15 & -0.19 & 0.044 & 0 \\
0 & 0.37 & 0 & -0.067 & 0 & 0.06 & 0 \\
0 & 0.38 & 0 & -0.068 & 0.31 & 0.061 & 0 \\
0 & -0.13 & 0.23 & 0 & 0 & -0.34 & -0.082 \\
0 & 0 & 0.5 & 0.15 & 0 & 0.045 & 0 \\
0 & 0 & -0.5 & -0.15 & -0.23 & -0.045 & 0 \\
-0.094 & -0.3 & 0 & 0 & 0 & -0.064 & 0 \\
0.19 & 0.28 & -0.2 & 0 & 0 & 0.06 & 0 \\
0.19 & 0.17 & 0 & 0.079 & 0 & 0.06 & 0 \\
0.095 & 0.3 & 0 & 0 & 0.18 & 0.065 & 0 \\
0.18 & 0.35 & -0.34 & -0.062 & 0 & 0.055 & 0 \\
0.19 & -0.37 & 0 & 0.17 & 0 & -0.028 & 0 \\
-0.095 & -0.4 & 0.56 & 0.18 & 0 & -0.03 & 0 \\
-0.19 & 0.39 & 0 & -0.18 & 0.46 & 0.029 & 0 \\
0.12 & -0.13 & 0 & 0 & 0 & -0.33 & -0.08 \\
-0.12 & 0.13 & 0 & 0 & 0.19 & 0.33 & 0.081 \\
-0.26 & 0 & 0 & -0.13 & 0 & 0 & 0.011 \\
0.15 & 0 & 0 & 0 & 0 & -0.28 & -0.075 \\
-0.16 & 0 & -0.25 & -0.15 & 0 & 0 & 0.012 \\
0.084 & 0 & 0.13 & 0 & 0 & -0.29 & -0.077 \\
-0.26 & 0 & 0 & -0.14 & 0.19 & 0 & 0.011 \\
0.15 & 0 & 0 & 0 & -0.11 & -0.28 & -0.075 \\
0 & -0.37 & 0 & 0.082 & 0 & 0 & 0.014 \\
0 & 0.31 & 0 & 0 & 0 & 0.25 & 0.049 \\
0 & 0.38 & 0 & -0.084 & 0.31 & 0 & -0.014 \\
0 & 0.31 & 0 & 0 & 0.26 & 0.25 & 0.049 \\
0 & 0.29 & 0.17 & 0 & 0 & 0.061 & 0 \\
0 & -0.29 & -0.17 & 0 & -0.32 & -0.062 & 0 \\
0 & -0.44 & 0.42 & 0.2 & 0 & -0.033 & 0 \\
0 & -0.44 & 0.42 & 0.2 & -0.17 & -0.033 & 0 \\
0 & 0 & -0.49 & -0.14 & 0 & 0 & 0.011 \\
0 & 0 & 0.28 & 0 & 0 & -0.28 & -0.075 \\
0 & 0 & 0.5 & 0.14 & 0.23 & 0 & -0.011 \\
0 & 0 & 0.28 & 0 & 0.13 & -0.29 & -0.075 \\
0.12 & 0.27 & 0 & 0 & 0 & 0 & -0.015 \\
0.19 & 0.26 & -0.16 & 0 & 0 & 0 & -0.014 \\
0.19 & 0.17 & 0 & 0.062 & 0 & 0 & -0.014 \\
-0.12 & -0.27 & 0 & 0 & -0.14 & 0 & 0.015 \\
0.15 & 0.14 & 0 & 0 & 0 & -0.18 & -0.056 \\
-0.18 & -0.34 & 0.33 & 0.077 & 0 & 0 & 0.013 \\
-0.15 & -0.29 & 0.29 & 0 & 0 & -0.24 & -0.046 \\
-0.19 & 0.37 & 0 & -0.18 & 0 & 0 & -0.0067 \\
0.095 & -0.19 & 0 & 0 & 0 & -0.33 & -0.077 \\
0.094 & 0.4 & -0.56 & -0.19 & 0 & 0 & -0.0071 \\
-0.046 & -0.19 & 0.27 & 0 & 0 & -0.34 & -0.078 \\
-0.19 & 0.39 & 0 & -0.19 & 0.46 & 0 & -0.007 \\
-0.096 & 0.19 & 0 & 0 & 0.23 & 0.34 & 0.078 \\
0 & 0.26 & 0.21 & 0 & 0 & 0 & -0.015 \\
0 & 0.26 & 0.21 & 0 & 0.32 & 0 & -0.015 \\
0 & -0.44 & 0.41 & 0.21 & 0 & 0 & 0.0078 \\
0 & 0.2 & -0.19 & 0 & 0 & 0.35 & 0.08 \\
0 & 0.44 & -0.42 & -0.21 & 0.17 & 0 & -0.0079 \\
0 & 0.2 & -0.19 & 0 & 0.076 & 0.35 & 0.08
\end{bmatrix}
\mathbf{z} +
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ -0.2 \\ 0.17 \\ 0.18 \\ 0 \\ 0 \\ 0 \\ -0.23 \\ -0.24 \\ 0.082 \\ 0 \\ 0 \\ 0.19 \\ -0.17 \\ -0.1 \\ -0.19 \\ -0.21 \\ 0.23 \\ 0.25 \\ -0.24 \\ 0.079 \\ -0.08 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.23 \\ -0.19 \\ -0.23 \\ -0.19 \\ -0.18 \\ 0.18 \\ 0.27 \\ 0.27 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.17 \\ -0.16 \\ -0.1 \\ 0.17 \\ -0.084 \\ 0.21 \\ 0.18 \\ -0.23 \\ 0.12 \\ -0.25 \\ 0.12 \\ -0.24 \\ -0.12 \\ -0.16 \\ -0.16 \\ 0.27 \\ -0.12 \\ -0.27 \\ -0.12
\end{bmatrix}
= \mathbf{0} \qquad (7.81)
$$

It is interesting to note that for example consistency relation 18 and 21 include the same set of sensors, even though these consistency relations are linearly independent. These two consistency relations corresponds to the MSS sets $\{e_1, e_2, e_3, e_4, e_5, e_7, e_8, e_{12}, e_{14}\}$ and $\{e_1, e_2, e_3, e_4, e_7, e_8, e_9, e_{12}, e_{14}\}$.

Static linear models fulfill Assumption 7.1 and Algorithm 7.1 can therefore be used. A future work will be to investigate which other types of models that fulfill Assumption 7.1. It is also interesting to find easily checked sufficient and if possible necessary conditions for Assumption 7.1 to hold.

7.3.3 Tree Model Assumption Cannot be Validated

When Assumption 7.1 cannot be validated for a diagnostic model $\mathbb{M}$, it is still probable that Assumption 7.1 holds for many models $M \subseteq^* \mathbb{M}$. Therefore one strategy is to find maximal tree models using the structure as before and then validate that no superset to each maximal tree model is a minimal rejectable model. Before an algorithm is presented that find a γ using this strategy, a theorem is presented that gives a sufficient analytical condition for a model M to conclude that no superset is a minimal rejectable model.

Theorem 7.8. *Assume that* M *is a model such that*

$$\forall \mathbf{z}\big(\mathrm{sol}(M, \mathbf{z}) \neq \varnothing \rightarrow (\exists e \in M : \mathrm{sol}(M \backslash \{e\}, \mathbf{z}) = \mathrm{sol}(M, \mathbf{z}))\big) \tag{7.82}$$

then no superset to M *is a minimal rejectable model.*

Proof. Assume that there is a minimal rejectable model $\hat{M}$ such that $M \subset \hat{M}$. From the definition of minimal rejectable model it follows that

$$\exists \mathbf{z}(\mathrm{sol}(\hat{M}, \mathbf{z}) = \varnothing \wedge \forall M' \subset \hat{M} : \mathrm{sol}(M', \mathbf{z}) \neq \varnothing) \tag{7.83}$$

Let $\mathbf{z} = \mathbf{z}_0$ fulfill (7.83). Since $M \subset \hat{M}$ it follows from (7.83) that

$$\mathrm{sol}(M, \mathbf{z}_0) \neq \varnothing \tag{7.84}$$

Expression (7.82) and (7.84) imply that there exists an $e \in M$ such that

$$\mathrm{sol}(M, \mathbf{z}_0) = \mathrm{sol}(M \backslash \{e\}, \mathbf{z}_0) \tag{7.85}$$

From the definition of operator sol, it follows that

$$\mathrm{sol}(\hat{M}, \mathbf{z}_0) = \mathrm{sol}(M, \mathbf{z}_0) \cap \mathrm{sol}(\hat{M} \backslash M, \mathbf{z}_0) \tag{7.86}$$

Using (7.85) and (7.86) it holds that

$$\mathrm{sol}(M, \mathbf{z}_0) \cap \mathrm{sol}(\hat{M} \backslash M, \mathbf{z}_0) = \mathrm{sol}(M \backslash \{e\}, \mathbf{z}_0) \cap \mathrm{sol}(\hat{M} \backslash M, \mathbf{z}_0) \tag{7.87}$$

From the definition of operator sol, it follows that

$$\mathrm{sol}(M \backslash \{e\}, \mathbf{z}_0) \cap \mathrm{sol}(\hat{M} \backslash M, \mathbf{z}_0) = \mathrm{sol}(\hat{M} \backslash \{e\}, \mathbf{z}_0) \tag{7.88}$$

From (7.86), (7.87) and (7.88) it follows that

$$\mathrm{sol}(\hat{M}, \mathbf{z}_0) = \mathrm{sol}(\hat{M} \backslash \{e\}, \mathbf{z}_0) \tag{7.89}$$

Now, since $\mathbf{z}_0$ satisfy (7.83) it follows that (7.83) and (7.89) is a contradiction. Hence $\hat{M}$ is not a minimal rejectable model. □

Theorem 7.8 will be illustrated in an example considering the same diagnostic model as in Example 7.7.

Example 7.9 The MSS models are

$$\text{mss}\mathbb{M} = \{\{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_3, e_4\}, \{e_5\}\} \tag{7.90}$$

Consider the first MSS $\{e_1, e_2, e_3\} = \{z_1 = x_1, x_1 = x_2^2, z_2 = x_2\}$. According to Theorem 7.8 we take an arbitrary $\mathbf{z}$ such that $\{e_1, e_2, e_3\}$ is fulfilled. Denote this observation $\hat{z}_1$ and $\hat{z}_2$. Then it holds that

$$\text{sol}(\{e_1, e_2, e_3\}, \mathbf{z}) = \{(x_1, x_2) | x_1 = \hat{z}_1, x_2 = \hat{z}_2\} \tag{7.91}$$

The e in (7.82) can be chosen as e_2 in this example. Then

$$\text{sol}(\{e_1, e_3\}, \mathbf{z}) = \{(x_1, x_2) | x_1 = \hat{z}_1, x_2 = \hat{z}_2\} \tag{7.92}$$

Hence from (7.82), (7.91), and (7.92) it follows that no superset to $\{e_1, e_2, e_3\}$ is a minimal rejectable model. In this example all MSS sets fulfill Theorem 7.8.

Now an algorithm is presented that finds a γ that corresponds to a sound and complete diagnostic system that uses the result of Theorem 7.8. The algorithm is rather computationally complex and therefore not suitable for large diagnostic models. Some details of how each step can be performed are omitted. Instead an example after the algorithm will show how each step is performed for a particular case. Let γ_b denote the set of models testing behavioral mode b.

Algorithm 7.2.
Input: $\mathbb{M}$.

For each $b \in \mathcal{B}$

a) *If* b *satisfies Theorem 6.12 then let* $C_b = M_b^+$, *else let* $C_b = M_b$.

b) *Set* $\gamma_b = \text{mss}C_b$.

c) *For each* $M \in \gamma_b$ *do the test according to Theorem 7.8. If (7.82) not holds then set* $\gamma_b = \gamma_b \setminus \{M\}$.

d) *Find all maximal models* $M' \subseteq C_b$ *that are connected and and that fulfill* $\forall M \in \gamma_b : M \not\subseteq M' \wedge M' \not\subseteq M)$. *Add all such models* M' *to* γ_b.

Output: $\gamma = \bigcup_{b \in \mathcal{B}} \gamma_b$

Next an example show how Algorithm 7.2 is applied to a model.

Example 7.10 Consider the diagnostic model $\mathbb{M}$ in Example 7.2. This model is not linear and therefore it is not clear that Assumption 7.1 is fulfilled. Then

Algorithm 7.2 can be applied to $\mathbb{M}$. In (a) the resulting behavioral models are calculated as in Example 6.8. The result is

b	C_{b_i}	$\text{mss}C_{b_i}$
b_1	$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_3, e_4\}$
b_2	$\{e_2, e_3, e_4\}$	$\{e_3, e_4\}$
b_3	$\{e_1, e_2, e_4\}$	$\{e_1, e_2, e_4\}$
b_4	$\{e_1, e_2, e_3, e_5\}$	$\{e_1, e_2, e_3\}, \{e_5\}$
b_5	$\emptyset$	
b_6	$\{e_5\}$	$\{e_5\}$
b_7	$\{e_1, e_2, e_5\}$	$\{e_5\}$
b_8	$\{e_5\}$	$\{e_5\}$

(7.93)

The MSS sets are

$$\text{mss}\mathbb{M} = \{\{e_5\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\} \tag{7.94}$$

In Example 7.9, the test in step (c) in Algorithm 7.2 is performed for each behavioral mode. Expression (7.82) is valid for all $M \in \gamma_b$.

To exemplify the calculations in step (d) the behavioral mode b_1 is used. For b_1 it holds that $C_{b_1} = M_{b_1} = \{e_1, e_2, e_3, e_4\}$. The MSS sets that are contained in M_{b_1} are $\{e_3, e_4\}$, $\{e_1, e_2, e_3\}$, and $\{e_1, e_2, e_4\}$. If all supersets of $\{e_3, e_4\}$ are built, it is clear that all sets with three equations are superset to some MSS set. If all subsets with two equations are found it is clear that all sets with two equations are subsets to some MSS set. Hence no model fulfills the condition in step (d) for behavioral mode b_1.

Another example is for behavioral mode b_7. From previous steps in Algorithm 7.2 it follows that $C_{b_7} = M_{b_7} = \{e_1, e_2, e_5\}$ and the only MSS sets is $\{e_5\}$. The maximal set that satisfies the conditions in step (d) is $\{e_1, e_2\}$. This set is added to γ.

After a great deal of calculations the output of Algorithm 7.2 is

$$\gamma = \{\{e_5\}, \{e_1, e_2\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\} \tag{7.95}$$

This set γ is equal to γ_m. The only analytical calculations done are the calculations done in Example 6.8 and in Example 7.9.

For large diagnostic models Algorithm 7.2 is computationally intractable. Next an algorithm is presented that reduce the computational complexity, but does not imply that the diagnostic system can be made sound. However, diagnostic systems produced using the next algorithm will often detect most inconsistencies and completeness is always obtained.

7.4 A Simplified Algorithm

Assume that Assumption 7.1 holds. Then we know that Algorithm 7.1 finds a γ that corresponds to a sound and complete diagnostic system. If we just do step (a) in Algorithm 7.1 then what result could we expect? If Corollary 7.6 holds then the result will be the same as when Algorithm 7.1 is applied. The diagnostic model in Example 7.8 fulfills Corollary 7.6. For many diagnostic models step (b) in Algorithm 7.1 is used little. For those models that step (b) is used little, the difference in results doing only step (a) will be small. If for some behavioral modes inconsistencies are important to detect, more detailed analysis according to previous sections can be done only for these behavioral modes.

Algorithm 7.3.
Input: $\mathbb{M}$
Set $\gamma := \text{mss}\mathbb{M}$.
Output: γ

Note that the output of this algorithm is not guaranteed to correspond to a sound diagnostic system but it always corresponds to a complete diagnostic system. Finally we use the same diagnostic models as in Example 7.10 to show that the output of Algorithm 7.3 that is much less computational complex than Algorithm 7.2 gives almost as good results.

Example 7.11 The structural analysis gives that the MSS sets are

$$\{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_3, e_4\}, \{e_5\} \tag{7.96}$$

Using Theorem 5.6 and the from previous examples calculated detection models, it can be seen that all behavioral modes except for b_7 satisfy (5.72). That means the corresponding diagnostic system is complete and all inconsistencies of all behavioral models except for b_7 can be detected by the diagnostic system. For b_7 inconsistencies are not detected for the following reason. No MSS set in $M_{b_7} = \{e_1, e_2, e_5\}$ is a superset to the detection model $\{e_1, e_2\}$. The observations when the candidates are different from the diagnoses is if $z_1 < 0$ and $z_3 = 0$. Then it follows that $b_7 \notin \mathcal{D}(\mathbf{z})$ since $\{e_1, e_2\}$ is not consistent with the observation, but the only assumption of a test that includes b_7 is the test for $\{e_5\}$. This set is consistent with the observation and therefore it follows that $b_7 \in \mathcal{C}(\mathbf{z})$. This shows that in this case the diagnostic system constructed using only the MSS sets is still rather good.

8

Structural Algorithms for Finding MSS Sets

In the previous chapter different algorithms were presented that under different assumptions found a set of models that corresponds to a sound and complete diagnostic system. In the simplified Algorithm 7.3, the only step was to find all MSS sets. In all other algorithms presented in Chapter 7 the MSS set finding was one of the most important steps. Hence to compute a "good" set of tests using the structure of a model, it is important to find the MSS sets. In this chapter we will describe algorithms that find all MSS sets for both differentiated-lumped structural-models DLSM:s and differentiated-separated structural models DSSM:s. The algorithms in this section take the structure of a diagnostic model as input . The structure of a diagnostic model can either be directly provided by the user or as explained in Section 3.3 obtained automatically from model equations. First in Section 8.1 different objectives for the MSS algorithm is presented and the basic steps toward finding MSS sets are described. Then in sections 8.2-8.7 all steps of the algorithm are presented. Finally in Section 8.8 some improvements of the computational complexity of the algorithm are discussed.

8.1 The Steps Toward Finding All MSS Sets

The objective is to find a set γ of all MSS sets that is contained in a diagnostic model $\mathbb{M}$. The algorithm can be summarized in the following steps.

Algorithm 8.1.
Input: The structure of a diagnostic model $\mathbb{M}$ *and let* $\mathtt{B} = \mathcal{B}$*.*

a) *Extracting a behavioral model: Let* $\mathtt{b} \in \mathtt{B}$ *and set* $\mathtt{B} = \mathtt{B} \backslash \{\mathtt{b}\}$*. Extract the* *behavioral model* $\mathsf{M_b}$. *First the no-fault model is extracted.*

b) *Differentiating the model: This step is done only if the algorithm is applied to a DSSM. Then equations that are meaningful to differentiate for finding MSS sets are structurally differentiated and added to the extracted model.*

c) *Extracting the overdetermined part of the model: Given the extracted model and the additional equations found in step (b), remove all equations that cannot be included in any MSS set, i.e. all equations not in the structurally overdetermined part.*

d) *Merging equation sets: To simplify the next step, merge sets of equations that have to be used together in each MSS set.*

e) *Finding MSS sets: Search for MSS sets in the resulting model from the previous step.*

f) *If* $\mathsf{B} \neq \varnothing$ *goto step (a).*

Output: All set of MSS sets.

When a diagnostic model is complex, for example contains a large number of equations, is highly redundant, and contains a large number of system behavioral modes the set γ will be large. Then the number of tests are too large to be able to check all tests on-line. Then it is interesting to find a subset of MSS sets that possibly has a desired isolability $\mathcal{I}_{\mathsf{d}}$. To find a subset of MSS sets given a desired isolability the following steps can be added to Algorithm 8.1.

The objective is to find a small set γ of MSS sets with, by the user defined desired isolability $\mathcal{I}_{\mathsf{d}}$. If full isolability is desired the resulting set γ has the same structural isolability obtained when $\mathsf{C_b} = \mathsf{M_b^+}$ in (6.122).

Algorithm 8.2.

g) *Evaluating isolability: Examine the isolability of the MSS sets found in step (e) in Algorithm 8.1.*

h) *Extracting behavioral models: If the isolability has to be improved to fulfill the desired isolability, a behavioral mode* $\mathsf{b} \in \mathsf{B}$ *is computed using the desired isolability and the so far obtained isolability. Then this behavioral mode is fed to step (a) in Algorithm 8.1. If no such* b *in* B *exists goto step (i).*

i) *Selecting a subset of MSS sets: Select a small set of MSS sets that contains the highest possible structural isolability or desired isolability.*

Output: A small set of MSS sets and their isolability.

Note that to avoid searching for all MSS sets in all behavioral models, Algorithm 8.2 has been organized so that, firstly, the fault free model is analyzed. The no-fault mode has often one of the most detailed models of all modes. Therefore this model contains many overdetermined models. Hence the MSS sets in $M_{\mathbf{NF}}$ will usually have a high isolability. Then, if it is necessary for achieving higher isolability, fault models are analyzed. Note also that step (i) in Algorithm 8.2 can be used separately. Any γ obtained from one of the algorithms 7.1, 7.2, or 7.3 can be used as the input to step (i) in Algorithm 8.2. The following sections discuss each of the steps in Algorithm 8.1 and Algorithm 8.2.

8.2 Extracting a Behavioral Model

The goal is to compute mss$\mathbb{M}$. As defined earlier mss$\mathbb{M}$ only contains the MSS sets that describe feasible models. One approach to find only such MSS sets is to search for MSS sets in each behavioral model. Then all MSS sets found will be feasible. Therefore the first step in Algorithm 8.1 extracts a behavioral model, in which MSS sets in later steps are to be found.

Algorithm 8.3.
Input: A diagnostic model $\mathbb{M}$ *and a system behavioral mode* $b \in \mathcal{B}$.

Calculate $M_b = \{e \in \mathbb{M} | b \in \text{ass}\, e\}$.

Output: $M_{\text{ext}} = M_b$.

An example will be used to exemplify the extraction.

Example 8.1 Consider the diagnostic model in Table 2.2 of the water-tank example in Section 2.2. If Algorithm 8.3 is applied to the behavioral mode **NF** the resulting model is $M_{\text{ext}} = M_{\mathbf{NF}} = \{e_1, e_3, e_4, e_7, e_{10}, e_{11}, e_{14}\}$.

To reduce the computational complexity of finding all MSS sets for all different behavioral modes, it is often a good choice to start with the no-fault model $M_{\mathbf{NF}}$. It is common that faulty behavioral models M_b are obtained by removing different equation sets from the no-fault model $M_{\mathbf{NF}}$. For these behavioral modes b it holds that $M_b \subseteq M_{\mathbf{NF}}$. Then no extra search for MSS sets needs to be performed for these behavioral models, because the MSS sets found in M_b will also be included in $M_{\mathbf{NF}}$. Hence all the MSS sets in a $M_b \subseteq M_{\mathbf{NF}}$ are found in the MSS sets of $M_{\mathbf{NF}}$.

Example 8.2 Consider the diagnostic model in Example 7.8. There are 15 model equations. The behavioral modes are described by removing different equation sets from $M_{\mathbf{NF}}$. If all multiple faults are considered the number of behavioral modes are $2^{15} = 32768$. Finding all MSS sets in a behavioral model is computationally complex as we will see later. This computational cost grows rapidly if the search for MSS sets has to be done in each behavioral mode individually. However, if all MSS sets first is found for the no-fault model, then it can be realized that all MSS sets for all behavioral models are obtained. This follows from the fact that all behavioral models M_b are subsets of the no-fault behavioral model, i.e. $M_b \subseteq M_{\mathbf{NF}}$. Then if an MSS set M is included in $M \subseteq M_b$ then $M \subseteq M_{\mathbf{NF}}$.

8.3 Handling Dynamic Models

If a DLSM is used, the algorithms in this section can be omitted and the extracted structural model M_{ext} can be directly fed to the algorithms in Section 8.4. If the model is a DSSM, derivatives have to be considered. Then there are two choices: either the additional structural information in a DSSM is used and then the method that will be described in Section 8.3.1 has to be used, or the DSSM is transformed to a DLSM that is described in Section 8.3.2.

8.3.1 Differentiating the Model

In this section an algorithm for handling derivatives is defined. This algorithm is referred to as Algorithm 8.4. First an example will show why differentiation has to be considered.

Example 8.3 Consider the model M_{ext} in Example 8.1. This model is a part of the model in Table 2.2. An algorithm that is not capable of differentiating equations can obviously not eliminate $\dot{h}$ in e_3, because there is no other equation including $\dot{h}$. In general, all derivatives of a model M have to be considered. If $M^{(i)}$ denote the set of the i:th time derivative of each element, the equation set generally considered is $\cup_{i=0}^{\infty} M^{(i)}$. □

To summarize the example, Algorithm 8.4 must be capable of differentiating equations. The next question to answer is if it is possible to predict the structural model of a differentiated analytical model by using only the structural model of the analytical diagnostic model? An example is used to answer this question.

Example 8.4 Consider again the model in Example 8.1. The differentiated equation $\dot{e}_4$ is $\dot{h} = 2 f_{out} \dot{f}_{out}$. The variable h is *linearly dependent* in e_4 and therefore $\dot{h}$ is linearly contained in equation $\dot{e}_4$. Furthermore, both f_{out} and $\dot{f}_{out}$ are nonlinearly contained in $\dot{e}_4$ as a consequence of the fact that f_{out} is nonlinearly contained in e_4.

This example shows that variables are handled in different ways depending on if they are linearly or nonlinearly dependent. To be able to take this different treatment into account, information about which variables that are linearly contained is added to the structural model. With this additional knowledge a structural differentiation can be defined that produce a correct structural representation of differentiated equations. Structural differentiation for an arbitrary variable x and an arbitrary equation e is defined in the following way:

1. If x is linearly contained in e then $\dot{x}$ is linearly contained in $\dot{e}$.

2. If x is nonlinearly contained in e then both x and $\dot{x}$ are nonlinearly contained in $\dot{e}$.

Now structural differentiation can be applied to the structural model. Since all numbers of differentiations of each equation implies a new equation, there are infinitely many equations in the differentiated model. Let $m(z)$ be a limit for variable $z \in Z$ of the order of derivative that can be considered as possible to estimate. If these limits are introduced and if the structural assumption to be presented next is fulfilled, it is possible to find all MSS sets in a finite subset of the differentiated model.

Before the assumption is presented, some notation is needed. If g is any equation, function or variable, let $g^{(i)}$ denote the i:th time derivative of g. Then define $\overline{\text{var}}_X E = \{\text{undifferentiated } x | \exists i (x^{(i)} \in \text{var}_X E)\}$, e.g. $\overline{\text{var}}_{X_u \cup Z}\{z = \dot{x}\} = \{z, x\}$.

Assumption 8.1. *The model* M_{ext} *has the property*

$$\forall E \subseteq M_{ext} : |E| \leq |\overline{\text{var}}_{X_u \cup Z} E|. \tag{8.1}$$

The meaning of condition (8.1) is that each subset of equations include more or equally many different variables, considering derivatives as the same variable. If Assumption 8.1 is not fulfilled and there are no redundant equations, the model would normally be inconsistent.

The easiest way to verify Assumption 8.1 is to verify the equivalent statement that there is a complete matching of M_{ext} into $\overline{\text{var}}_{X_u \cup Z} M_{ext}$ in the corresponding bipartite graph $\mathcal{G}(M_{ext}, \overline{\text{var}}_{X_u \cup Z} M_{ext})$.

A sufficient condition that there is a model with finitely many equations that contains all MSS sets is that the model M_{ext} satisfy Assumption 8.1 and all known variables have finite limitations.

Algorithm 8.4 is greatly influenced by Pantelides' algorithm (Pantelides 1988). Before the algorithm is presented, a few definitions are introduced. Let $\bar{M} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{\alpha_i} \{e_i^{(j)}\}$ be a differentiated model of $M = \bigcup_{i=1}^{n} \{e_i\}$. Then the highest number of differentiations in $\bar{M}$ of equation i is α_i. Let $M^{max} = \{e_i^{(\alpha_i)} | 1 \leq i \leq n\}$ be the set of *most differentiated equations* in M and let $M^{\infty} = \{e_i^{(j)} | e_i \in M, j \in \mathbb{N}\}$. The *highest derivative* of a non-differentiated variable x in a model M is denoted $\beta(M, x)$, i.e. $\beta(M, x) = \max(\{i | x^{(i)} \in \text{var}_{X_u \cup Z} M\})$. Finally let $\widehat{\text{var}} M$ be the variables $\text{var}_{X_u \cup Z} M$ that fulfill the following two requirements:

- It is the highest derivative of each variable that are considered.
- It is the variables, whose derivative is unknown.

For example, if $\dot{z} \in \text{var}_{X_u \cup Z} M$, $\forall i \in \mathbb{Z}_+ \backslash \{1\} : z^{(i)} \notin \text{var}_{X_u \cup Z} M$, and $m(z) = 1$, then $\dot{z} \in \widehat{\text{var}} M$ because $\dot{z}$ is the highest derivative of z in M and $\ddot{z}$ is unknown.

Algorithm 8.4.
Input: The extracted model M_{ext}*, a description of which variables that are linearly contained in each equation, and for each* $z \in \overline{\text{var}}_Z M_{ext}$*,* $m(z) < \infty$*.*

1. *Let the current model* M_c *be* M_{ext} *and let* $i = 1$*.*
2. *If* $i \leq |M_{ext}|$ *then let* M_c^{max} *be only the most differentiated equations of* M_c*. Let* $M_c^{max}(i)$ *denote the* i *first equations in* M_c^{max} *and let equation* i *in*

M_c^{max} *be denoted* e_i*. A complete matching of* $M_c^{max}(i-1)$ *into* $\widehat{\text{var}}M_c^{max}$ *in the bipartite graph* $\mathcal{G}(M_c^{max}, \widehat{\text{var}}M_c^{max})$ *is found in previous steps. Search for an augmented path in* $\mathcal{G}(M_c^{max}, \widehat{\text{var}}M_c^{max})$ *from* e_i *to an unassigned variable in* $\widehat{\text{var}}M_c^{max}$*.*

a) *If an augmented path* P *is found with respect to the matching* Γ *a complete matching of* $M_c^{max}(i)$ *into* $\widehat{\text{var}}M_c^{max}$ *is* $(\Gamma \cup P)\backslash(\Gamma \cap P)$*. Set* $i = i + 1$ *and goto step (2).*

b) *No augmented path is found. Then an MSS set with respect to* $\widehat{\text{var}}M_c^{max}$ *is found as follows. Let all edges not included in the matching be directed edges from the equation vertices to the variable vertices. Then the MSS set with respect to* $\widehat{\text{var}}M_c^{max}$ *is defined as all equation vertices* reachable *from* e_i*. Denote this MSS set* E*. Note the difference between this set which is MSS with respect to* $\widehat{\text{var}}M_c^{max}$ *instead of MSS with respect to* X_u*. Differentiate* E *until* $|\widehat{\text{var}}E^{(i)}| \geq |E|$ *using the description of which variables that are linearly contained. Let the obtained differentiated model be* M_c*. Goto step (2).*

3. *Rename the current model* M_c *to* M_{diff}*.*

Output: M_{diff}*.*

If $\text{mss}_{\text{all}}M = \text{mss}(\cup_{i=0}^{\infty} M^{(i)})$ then it is possible to state the following theorem.

Theorem 8.1. *If Assumption 8.1 is satisfied and for each* $z \in \overline{\text{var}}_Z M_{ext}$*,* $m(z) < \infty$*, then*

$$\text{mss}_{\text{all}} M_{ext} = \text{mss} M_{diff}$$

The consequence of this theorem is that all MSS sets that are possible to find if the model M_{ext} is differentiated an infinite number of times, can always be found in M_{diff}. Before Theorem 8.1 is proved the continuation of Example 8.1 is presented to describe how Algorithm 8.4 works.

Example 8.5 Consider the structural model in Table 2.2 and the extracted model $M_{ext} = M_{\mathbf{NF}} = \{e_1, e_3, e_4, e_7, e_{10}, e_{11}, e_{14}\}$. The corresponding bipartite graph
$\mathcal{G}(M_{ext}, \text{var}_{X_u \cup Z} M_{ext})$ is shown in Figure 8.1. This model satisfies Assumption 8.1 since it is possible to find a complete matching of M_{ext} into $\overline{\text{var}}_{X_u \cup Z} M_{ext}$ in the graph $\mathcal{G}(M_{ext}, \overline{\text{var}}_{X_u \cup Z} M_{ext})$ shown in Figure 8.2. An example of a complete matching of M_{ext} into $\overline{\text{var}}_{X_u \cup Z} M_{ext}$ is $\{e_1, f_{in}\}$, $\{e_3, f_{out}\}$, $\{e_4, h\}$, $\{e_7, f\}$, $\{e_{10}, y_h\}$, $\{e_{11}, f_{int}\}$, and $\{e_{14}, y_f\}$. This matching is shown in Figure 8.2 by the bold edges. Let $m(u) = m(y_f) = 1$ and $m(y_h) = 0$. According to Theorem 8.1 it is possible to use Algorithm 8.4 to produce the model M_{diff}.

Step (2) in Algorithm 8.4 is fed with the structural model M_c shown in Figure 8.1 and the m-values. Figure 8.3 shows the graph $\mathcal{G}(M_c^{max}, \widehat{\text{var}}M_c^{max})$ built in step (2). Note that the vertex corresponding to h is not considered, because h is not the highest derivative of h in the model. The known variables u and y_f

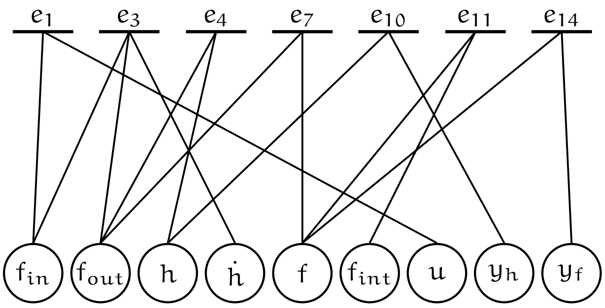

Figure 8.1 *The bipartite graph* $\mathcal{G}(M_{ext}, \text{var}_{X_u \cup Z} M_{ext})$.

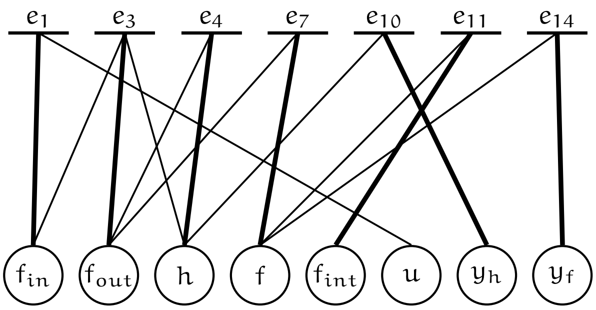

Figure 8.2 *The bipartite graph* $\mathcal{G}(M_{ext}, \overline{\text{var}}_{X_u \cup Z} M_{ext})$. *The thick edges denotes a complete matching of* M_{ext} *into* $\overline{\text{var}}_{X_u \cup Z} M_{ext}$.

have known derivatives and are therefore not included in $\widehat{\text{var}} M_c^{max}$. However, the derivative of y_h is an unknown variable and y_h is therefore included.

Step (2) in Algorithm 8.4 searches for an augmented path from e_1 to $\widehat{\text{var}} M_c^{max}$ in the graph showed in Figure 8.3. The path $P = \{\{e_1, f_{in}\}\}$ is found and this single edge becomes the first matching, i.e. $\Gamma = \{\{e_1, f_{in}\}\}$. Then the assignment $\{e_3, f_{out}\}$ will be found and $\Gamma := \{\{e_1, f_{in}\}, \{e_3, f_{out}\}\}$. When e_4 is analyzed the alternating path found is $P := \{\{e_4, f_{out}\}, \{f_{out}, e_3\}, \{e_3, \dot{h}\}\}$. The new matching becomes

$$\begin{aligned} \Gamma &:= (\Gamma \cup P) \setminus (\Gamma \cap P) = \{\{e_1, f_{in}\}, \{e_3, f_{out}\}, \{e_4, f_{out}\}, \{e_3, \dot{h}\}\} \setminus \{\{e_3, f_{out}\}\} = \\ &= \{\{e_1, f_{in}\}, \{e_4, f_{out}\}, \{e_3, \dot{h}\}\} \end{aligned}$$

The assignments in the matching are then found in the following order $\{e_7, f\}$, $\{e_{10}, y_h\}$, and $\{e_{11}, f_{int}\}$. These are the edges denoted with a bold edge in the graph in Figure 8.3. When e_{14} is going to be assigned, there is no variable vertex left. Since no augmenting path is found, step (2b) finds an MSS set with respect to $\widehat{\text{var}} M_c^{max}$. When edges not contained in the matching are directed from equation

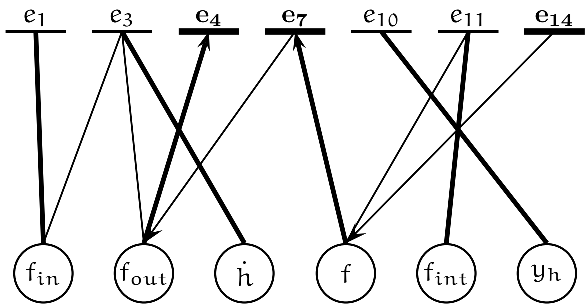

Figure 8.3 *The bipartite graph $\mathcal{G}(M_c^{max}, \widehat{\text{var}}M_c^{max})$ built in step (2). The bold edges are the matching found. The bold equation vertices are the MSS set found in step (2b).*

vertices to variable vertices, the reachable equation vertices from e_{14} are e_4 and e_7. A directed path from e_{14} containing equation e_4 and e_7 are shown in Figure 8.3 by arrows on the edges. This path proves that e_4 and e_7 is reachable form e_{14} considering the directed edges. Hence this is the equation set to be differentiated.

The structural differentiation uses additional information about which variables that are nonlinearly included in each equation. Nonlinearly included variables are denoted with "O" in Table 8.1. Differentiating once implies that $\dot{y}_f$ appears in $\widehat{\text{var}}(\{\dot{e}_4, \dot{e}_7, \dot{e}_{14}\})$. The new model consists of $\{e_1, e_3, e_4, \dot{e}_4, e_7, \dot{e}_7, e_{10}, e_{11}, e_{14}, \dot{e}_{14}\}$ and the new bipartite graph showed in Figure 8.4 is extracted in step (2). Equations e_4, e_7, and e_{14} are not anymore the most differentiated equations in the new model. Further, $\dot{y}_f$ is included, because $\ddot{y}_f$ is considered as an unknown variable. Note that an edge in the matching in Figure 8.3 is either unchanged or replaced with an edge between the replaced vertices corresponding to the differentiated equation and the differentiated variable in Figure 8.4. For example $\{e_1, f_{in}\}$ is unchanged and the edge $\{e_4, f_{out}\}$ in Figure 8.3 is replaced with $\{\dot{e}_4, \dot{f}_{out}\}$ in Figure 8.4.

Step (2) finds an assignment for $\dot{e}_{14}$. The structural model M_{diff} obtained from Algorithm 8.4 is shown in Table 8.1. The three differentiated equations are $\dot{e}_4 : \dot{h} = 2\, f_{out}\, \dot{f}_{out}$, $\dot{e}_7 : \dot{f} = \dot{f}_{out}$, and $\dot{e}_{14} : \dot{y}_f = \dot{f}$. Note the exact correspondence between the analytical differentiation and the structural differentiation in Table 8.1.

Now, we finally prove Theorem 8.1.

Proof. The proof consists of two parts. The first part states that Algorithm 8.4 terminates and that the differentiated model has the property that there is a complete matching from M_{diff}^{max} into $\widehat{\text{var}}M_{diff}^{max}$. In the second part this complete matching is used to show that $\text{mss}_{\text{all}}M_{ext} = \text{mss}M_{diff}$.

Algorithm 8.4 terminates when $i = |M_{ext}|$. The variable i is increased in step (2a). Step (2a) is done when an augmented path from e_i to an unassigned

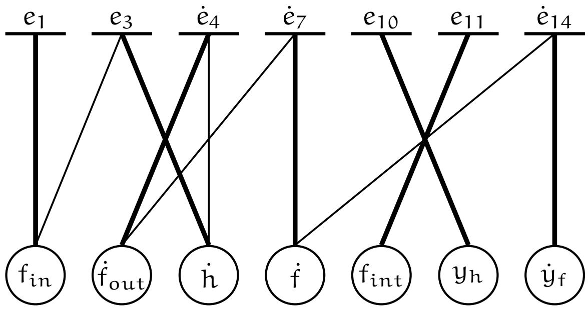

Figure 8.4 *The bipartite graph $\mathcal{G}(M_c^{max}, \widehat{\text{var}}M_c^{max})$ built in step (2) after one differentiation. The bold edges are the complete matching found in step (2).*

Table 8.1 *The structural model M_{diff}^{DSSM} obtained from Algorithm 8.4 when applied to the structural model $M_{ext} = M_{\mathbf{NF}}^{DSSM}$. Nonlinearly included variables are denoted with an "O".*

model	unknown								known			
	f_{in}	f_{out}	$\dot{f}_{out}$	h	$\dot{h}$	f	$\dot{f}$	f_{int}	u	y_h	y_f	$\dot{y}_f$
e_1	X								X			
e_3	X	X			X							
e_4		O		X								
$\dot{e}_4$		O	O		X							
e_7		X				X						
$\dot{e}_7$			X				X					
e_{10}				X						X		
e_{11}						X		X				
e_{14}						X					X	
$\dot{e}_{14}$							X					X

variable in $\widehat{\text{var}}M_c^{max}$ is found.

To show that Algorithm 8.4 terminates is equivalent to show that an augmented path from e_i to an unassigned variable in $\widehat{\text{var}}M_c^{max}$ is always found in finitely many iterations using step (2b).

First it will be shown that the differentiation in step (2b) is always terminated. Therefore assume that there is no augmented path from e_i to an unassigned variable in $\widehat{\text{var}}M_c^{max}$ in step (2). Then step (2b) finds an MSS set with respect to $\widehat{\text{var}}M_c^{max}$. Let this MSS set be denoted E. In Figure 8.5 the vertices denoted with lines are the equation vertices M_c^{max} and the circles denotes the variable vertices $\widehat{\text{var}}M_c^{max}$. The bold edges represents a matching. The bold equation vertices are an MSS set E with respect to $\widehat{\text{var}}M_c^{max}$.

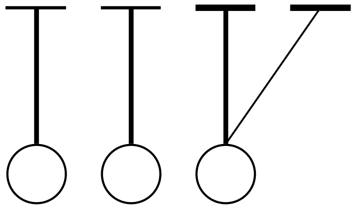

Figure 8.5 *The equation vertices are M_c^{max} and the variable vertices are $\widehat{\text{var}} M_c^{max}$. The bold edges represents a matching. The bold equation vertices are an MSS set E with respect to $\widehat{\text{var}} M_c^{max}$.*

Since both the highest derivatives of each variable and all limits on known variables are finite, it is possible to exceed those limits by differentiating E, let say m number of times. Then it can be realized that the model M_c and the MSS set E have the following property:

$$\begin{aligned}&\big(\forall x \in \overline{\text{var}}_{X_u \cup Z} E : \beta(M_c, x) < \infty \wedge \\ &\forall z \in \overline{\text{var}}_Z E : m(z) < \infty\big) \quad \Rightarrow \\ &\exists m \in \mathbb{N} : \big(\forall x \in \overline{\text{var}}_{X_u} E : \beta(E^{(m)}, x) \geq \beta(M_c, x) \wedge \\ &\forall z \in \overline{\text{var}}_Z E : \beta(E^{(m)}, z) \geq \max(\beta(M_c, z), m(z))\big)\end{aligned} \tag{8.2}$$

Assumption 8.1 guarantees that $|E| \leq |\overline{\text{var}}_{X_u \cup Z} E|$. According to expression (8.2) each variable in $\overline{\text{var}}_{X_u \cup Z} E$ will have a corresponding derivative in $\widehat{\text{var}} E^{(m)}$. Hence $|\widehat{\text{var}} E^{(m)}| = |\overline{\text{var}}_{X_u \cup Z} E| \geq |E|$ which is the stop condition of step (2b). After the redefinition of M_c^{max} in step (2) at least one new variable is included in $\text{var}_{\widehat{\text{var}} M_c^{max}}(E^{(m)})$. According to Lemma 8.2 the differentiation in step (2b) will not remove any corresponding edge in previous found matching.

Next it is shown that the loop using step (2b) terminates, i.e. after a finite number iterations using step (2b) Algorithm 8.4 finds an augmented path and step (2a) is applied. Since the previous matching has a corresponding matching, the corresponding matching together with the augmented path defines a new extended matching.

As explained above the differentiation is terminated and there is at least one new variable included in $\text{var}_{\widehat{\text{var}} M_c^{max}}(E^{(m)})$. The result of finding new variable vertices is divided into two cases.

1. All new variable vertices are already included in the matching. An example is shown in Figure 8.6 where the dashed edge is the newly appeared. Then there is a new MSS set $\hat{E}$ with respect to $\widehat{\text{var}} M_c^{max}$ in the right graph in Figure 8.6 denoted with bold vertices. Let the notation $x^{(i)}$ be generalized such that $\{x_1, \ldots, x_n\}^{(i)}$ denotes the set of variables $\{x_1^{(i)}, \ldots, x_n^{(i)}\}$. Then the definition of structural differentiation implies that the graphs $\mathcal{G}(E, \widehat{\text{var}} E)$

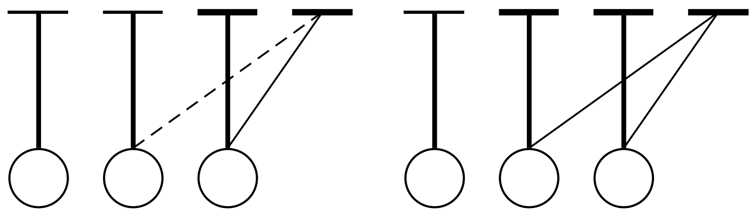

Figure 8.6 *In the left graph the two rightmost equations have been differentiated. As a result of the differentiation the dashed edge appears. The new variable vertex is already included in the matching as shown in the left graph. Then there is a new MSS set with respect to* $\widehat{\mathrm{var}}M_c^{max}$ *in the right graph denoted with bold vertices. Note that the MSS set in the left graph is a subset to the MSS set in the right graph.*

and $\mathcal{G}(E^{(m)}, (\widehat{\mathrm{var}}E)^{(m)})$ are *isomorphic* (Grimaldi 1994). If E is differentiated m number of times then it is clear according to how the MSS set is obtained in Algorithm 8.4 and the fact that the subgraphs $\mathcal{G}(E, \widehat{\mathrm{var}}E)$ and $\mathcal{G}(E^{(m)}, (\widehat{\mathrm{var}}E)^{(m)})$ are isomorphic that $E^{(m)} \subset \hat{E}$. Since the new MSS set $\hat{E}$ is including $E^{(m)}$ then this case can only be repeated i times. Therefore it is sufficient to prove that given case 2 an augmented path will be found and hence step (2b) will be followed by step (2a).

2. There is a new variable that is not included in the matching. All vertices are reachable from $e_i^{(m)}$ when all edges not included in the matching are directed edges from the equation vertices to the variable vertices, i.e. there is an augmented path from $e_i^{(m)}$ to the new variable vertex in $\widehat{\mathrm{var}}E^{(m)}$. This augmented path defines a new complete matching including $e_i^{(m)}$. In Figure 8.7 there is a new edge to a new unassigned variable. There is an augmenting path from the last equation vertex to the last variable vertex. In the right figure the new matching is defined.

Hence the algorithm will terminate and find a complete matching of M_{diff}^{max} into $\widehat{\mathrm{var}}M_{diff}^{max}$. Now it remains to prove that M_{diff} contains all MSS sets. From Lemma 8.3 it follows that $\mathrm{mss_{all}}M_{ext} \subseteq \mathrm{mss}M_{diff}$. Since $M_{diff} \subset M_{ext}^{\infty}$, it implies that $\mathrm{mss}M_{diff} \subseteq \mathrm{mss}M_{ext}^{\infty} = \mathrm{mss_{all}}M_{ext}$. Hence $\mathrm{mss}M_{diff} = \mathrm{mss_{all}}M_{ext}$. □

Now the two Lemmas 8.2 and 8.3 will be discussed and proven. According to Algorithm 8.4 assignments for each equation in M_c are sequentially found. However, it is important that the differentiation of the MSS set E in step (2b), does not remove any edge that is included in the matching. To show that differentiation does not remove any edge included in the matching, let a bipartite graph be defined

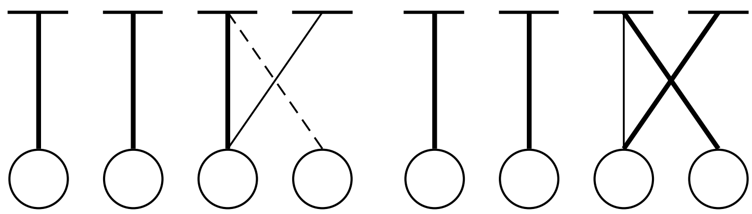

Figure 8.7 *In the left graph the to last equation vertices have been differentiated. As a result of the differentiation the dashed edge appears. There is an augmenting path from the last equation vertex to the last variable vertex. In the right figure the new matching is defined switching assigned and unassigned edges.*

as $\mathcal{G}_1 = \mathcal{G}(M_c^{max}, \widehat{\mathrm{var}} M_c^{max})$. Consider two subsequent graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, i.e. $\mathcal{G}_2$ is the resultant bipartite graph in step (2) after step (2b) is applied to $\mathcal{G}_1$. Suppose that step (2b) differentiate E, m number of times.

Lemma 8.2. *Step (2b) preserves any matching, i.e. if there is an edge* $\{e, x\}$ *included in the matching in* $\mathcal{G}_1$*, then either* $\{e, x\}$ *is included in* $\mathcal{G}_2$ *or there is a corresponding edge in* $\mathcal{G}_2$ *between* $\{e^{(m)}, x^{(m)}\}$.

Proof. Let e_i denote the first equation in $\mathcal{G}_1$ that has not been assigned. Assume that there is no augmenting path in $\mathcal{G}_1$ from e_i. Otherwise no differentiation has to be done and therefore it is nothing to prove. The complete matching of $M_c^{max}(i-1)$ into $\widehat{\mathrm{var}} M_c^{max}$ in $\mathcal{G}_1$ together with e_i define an MSS set E. Now for e it holds that either $e \notin E$ or $e \in E$.

1. If $e \notin E$, e is not differentiated. Then according to Algorithm 8.4 there can not be a directed path from e_i to e considering unmatched edges as directed edges from equation vertices to variable vertices. This implies that there is no directed path from e_i to x either. If any of the equations reachable from e_i included x then x would also be reachable from e_i. This is not true and the conclusion is that no equations including x are differentiated. Hence the variable x is not involved in the differentiation. The edge $\{e, x\}$ and the vertices e and x are unchanged from $\mathcal{G}_1$ to $\mathcal{G}_2$.

2. If $e \in E$, e is differentiated. Then according to Algorithm 8.4 there is a directed path from e_i to e considering unmatched edges as directed edges from equation vertices to variable vertices. Since the only incoming edge to e is the edge $\{e, x\}$ the only possible directed path to e goes through the variable vertex corresponding to x. Hence e and x are replaced with $e^{(1)}$ and $x^{(1)}$ in $\mathcal{G}_2$ respectively. The edge $\{e^{(1)}, x^{(1)}\}$ is obviously included in the graph corresponding to one differentiation. If the differentiation is repeated it

follows that $\{e^{(j)}, x^{(j)}\}$ will be included in the corresponding graph. Especially $\{e^{(m)}, x^{(m)}\}$ will be included in $\mathcal{G}_2$.

In both two cases the matching is preserved and therefore the lemma is proven. □

Lemma 8.3. *If there is a complete matching of the most differentiated equations in* M_{diff} *into the variable vertices in to* $\widehat{\text{var}}M_{diff}^{max}$*. Then all* $\text{mss}_{\text{all}}M_{ext} \subseteq \text{mss}M_{diff}$.

Proof. Let the equations and variables in the complete matching be denoted e_i and x_i respectively such that $\{e_i, x_i\}$ is an assignment. It is clear that $\forall j \in \mathbb{Z}^+ : e_i^{(j)} \notin M_{diff}$ and $\forall j \in \mathbb{Z}^+ : x_i^{(j)} \notin \text{var}_{X_u \cup Z} M_{diff}$.

Take an arbitrary set of equations E such that $E \subset M_{ext}^{\infty}$ and $E \cap (M_{ext}^{\infty} \setminus M_{diff}) \neq \varnothing$. Call this intersection E'. Let the equations in E' be

$$\begin{array}{l} e_1^{(\alpha_1)}, \cdots, e_1^{(\alpha_{n_1})} \\ \vdots \\ e_m^{(\alpha_1)}, \cdots, e_m^{(\alpha_{n_m})} \end{array} \tag{8.3}$$

Note that all $\alpha_i > 0$. According to the complete matching, it is clear that $x_i^{(\alpha)} \in \text{var}_{X_u \cup Z} e_i^{(\alpha)}$ for $0 < \alpha \leq \alpha_{n_i}$. Further $x_i^{(\alpha)} \notin M_{diff}$, where $0 < \alpha \leq \alpha_{n_i}$.

Now, the idea is to apply Lemma 3.5 on the variables set $X = \{x_i^{(\alpha_j)} | 1 \leq i \leq m, 1 \leq j \leq n_i\}$. The number of variables is $|X| = \sum_{i=1}^{m} n_i$. From the fact that $\text{var}_X M_{diff} = \varnothing$ and $x_i^{(\alpha)} \in varXe_i^{(\alpha)}$ it follows that $\text{equ}_E(X) = E'$. The number of equations in E' is $|E'| = \sum_{i=1}^{m} n_i = |X|$. Lemma 3.5 conclude that E can not be an MSS set. Hence, given any MSS set E, it follows that $E \subseteq M_{diff}$. □

8.3.2 Computing the DLSM

Another approach is to consider the differentiated-lumped structural-model. Using this approach, a smaller structural model than the differentiated structural-model is obtained. Hence this approach is especially suitable for large models. For the DLSM the limitations of the number of derivatives of known variables $m(z)$ are not needed.

The DLSM of $M_{\mathbf{NF}}$ obtained from Table 2.2 can be seen in Table 8.2. Note that the number of equations and unknown variables in the DLSM are 7 and 5 respectively. These numbers are smaller than corresponding numbers of the differentiated DSSM M_{diff}^{DSSM} in Table 8.1 with 10 equations and 8 unknown variables.

8.4 Simplifying the Model

It is a complex task to find all MSS sets in a structural model. Therefore it can be of great help if it is possible to simplify the model. Here two kinds of simplifications are used. In the first step the structurally overdetermined part of the model is found and in the second step equations that must be used together in MSS sets are merged.

Table 8.2 *The DLSM* $M_{\mathbf{NF}}^{\mathsf{DLSM}}$.

equation	unknown					known		
	f_{in}	f_{out}	h	f	f_{int}	u	y_h	y_f
e_1	X					X		
e_3	X	X	X					
e_4		X	X					
e_7		X		X				
e_{10}			X				X	
e_{11}				X	X			
e_{14}				X				X

8.4.1 Extracting the Overdetermined Part of the Model

In a first step, the structurally overdetermined part of $M_{\mathsf{diff}}^{\mathsf{DSSM}}$ or $M_{\mathbf{NF}}^{\mathsf{DLSM}}$ is found. This is done using canonical decomposition described in Section 3.4.2. As a reminder the description of canonical decomposition is rewritten. Canonical decomposition divides a model M into three parts: one structurally overdetermined denoted M^+, one structurally just-determined M^0 and one structurally underdetermined part M^-. This is accomplished by first finding a maximal matching in the bipartite graph $\mathcal{G}(M, \mathrm{var}_{X_u} M)$. Denote the assigned equations and variables in the maximal matching with M_m and X_m respectively. Now, all vertices such that there is an alternating path from $M \backslash M_m$ is the structurally overdetermined part of the model M^+. The structurally underdetermined part of the model M^- is the vertices such that there is an alternating path from $\mathrm{var}_{X_u}(M) \backslash X_m$. The remaining part of the model is the structurally just-determined part M^0. The output of this step will be denoted M_{over} and is equal to M_{diff}^+. The two different types of structural models of the water-tank example are used to illustrate this step.

Example 8.6 First we study the DLSM $M_{\mathbf{NF}}^{\mathsf{DLSM}}$ in Table 8.2. A maximal matching in the graph $\mathcal{G}(M_{\mathbf{NF}}^{\mathsf{DLSM}}, \mathrm{var}_{X_u} M_{\mathbf{NF}}^{\mathsf{DLSM}})$ is shown with bold crosses in Table 8.3 and with bold edges in Figure 8.8.

There exists an alternating path from the equation vertices not assigned, i.e. e_1 and e_{14}, to e_3, e_4, e_7, and e_{10}. An example of two alternating paths are $\{e_{14}, f\}$, $\{f, e_7\}$, $\{e_7, f_{out}\}$, $\{f_{out}, e_4\}$, $\{e_4, h\}$, $\{h, e_{10}\}$ and $\{e_1, f_{in}\}$, $\{f_{in}, e_3\}$. However it is not possible to find an alternating path between e_{11} and e_1 or e_{14}. This means that $(M_{\mathbf{NF}}^{\mathsf{DLSM}})^+ = M_{\mathsf{over}}^{\mathsf{DLSM}} = \{e_1, e_3, e_4, e_7, e_{10}, e_{14}\}$ and $(M_{\mathbf{NF}}^{\mathsf{DLSM}})^0 = \{e_{11}\}$.

For the differentiated DSSM in Table 8.1 the decomposition is shown in Table 8.4. As for the DLSM equation e_{11} is the structurally just-overdetermined part while the set of the remaining equations are the structurally overdetermined part.

Table 8.3 *A maximal matching in $\mathcal{G}(M_{\mathbf{NF}}^{\mathrm{DLSM}}, \mathrm{var}_{X_u} M_{\mathbf{NF}}^{\mathrm{DLSM}})$ is shown with bold crosses. Equation e_{11} is the structurally just-determined part and the set of all other equations is the structurally overdetermined part.*

model	unknown					known		
	f_{int}	f	h	f_{out}	f_{in}	u	y_h	y_f
e_{11}	**X**	X						
e_{14}		X						X
e_7		**X**		X				
e_{10}			**X**				X	
e_4			X	**X**				
e_3			X	X	**X**			
e_1					X	X		

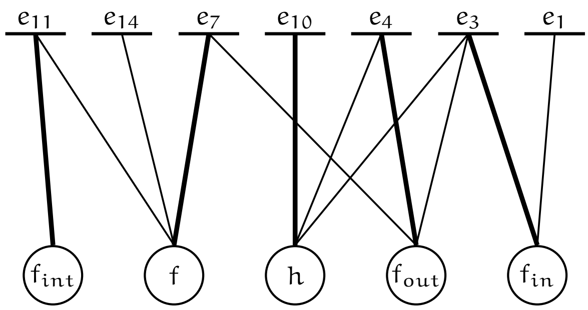

Figure 8.8 *The bipartite graph $\mathcal{G}(M_{\mathbf{NF}}^{\mathrm{DLSM}}, \mathrm{var}_{X_u} M_{\mathbf{NF}}^{\mathrm{DLSM}})$. The thick edges denotes a maximal matching in $\mathcal{G}(M_{\mathbf{NF}}^{\mathrm{DLSM}}, \mathrm{var}_{X_u} M_{\mathbf{NF}}^{\mathrm{DLSM}})$.*

8.4.2 Merging Equation Sets

In a second simplification step, equations that must be used together in MSS sets are merged. The rest of this section will be devoted to a discussion about this second step.

If there is a set $X \subseteq X_u$ with the property $1 + |X| = |\mathrm{equ}_{M_{over}}(X)|$, then either no equation in $\mathrm{equ}_{M_{over}}(X)$ or all equations in $\mathrm{equ}_{M_{over}}(X)$ must be used in MSS sets to fulfill the condition in Lemma 3.5. This means that all MSS sets including any equation of $\mathrm{equ}_{M_{over}}(X)$ have to include all equations in $\mathrm{equ}_{M_{over}}(X)$. The idea is to find these sets of equations. Then it is possible to merge $\mathrm{equ}_{M_{over}}(X)$ and only consider the unknown variables $\mathrm{var}_{X_u}(\mathrm{equ}_{M_{over}}(X)) \setminus X$.

This second simplification step finds subsets of variables that are included in exactly one more equation than the number of variables. To reduce the compu-

Table 8.4 *The structural model* $M_{\mathrm{diff}}^{\mathrm{DSSM}}$ *obtained from Algorithm 8.4 when applied to the structural model* M_{ext}. *The structural model* $M_{\mathrm{over}}^{\mathrm{DSSM}}$ *is all equations except* e_{11}.

model	unknown								known			
	f_{int}	f	f_{out}	h	$\dot{f}$	$\dot{f}_{out}$	$\dot{h}$	f_{in}	u	y_h	y_f	$\dot{y}_f$
e_{11}	X	X										
e_{14}		X									X	
e_7		X	X									
e_4			O	X								
e_{10}				X						X		
$\dot{e}_{14}$					X							X
$\dot{e}_7$					X	X						
$\dot{e}_4$			O			O	X					
e_3			X				X	X				
e_1								X	X			

tational complexity, a complete search for such sets is in fact not performed here. Instead only a search for single variables included in two equations is done. When a variable is included in just two equations, these equations are merged. If all variables are examined and some simplification was possible, then all remaining variables are examined once more. When no more simplifications can be made, the simplification step is finished and the resulting structural model is denoted M_{simp}. Note that with this strategy larger sets than two equations will also be merged, since the algorithm can merge sets of equations merged in previous steps.

Algorithm 8.5.
Input: M_{over}

1. *Set* $X = \mathrm{var}_{X_u} M_{\mathrm{over}}$ *and* $M_{\mathrm{simp}} = M_{\mathrm{over}}$.

2. *For all variables* $x \in X$ *do step (3).*

3. *If* $|\mathrm{equ}_{M_{\mathrm{over}}}(x)| = 2$ *then set* $X = X \backslash \{x\}$ *and let the two equations* $\mathrm{equ}_{M_{\mathrm{over}}}(x)$ *in* M_{simp} *be replaced with one new model* M *where* $\mathrm{var}_{X_u \cup Z} M = \mathrm{var}_{X \cup Z}(\mathrm{equ}_{M_{\mathrm{simp}}}(x))$. *For all* $e \in M_{\mathrm{simp}} \backslash \mathrm{equ}_{M_{\mathrm{simp}}}(x)$ *let* $\mathrm{var}_{X_u \cup Z} e = \mathrm{var}_{X \cup Z} e$.

4. *If some simplifications were made in step (3) go back to step (2).*

Output: M_{simp}.

The complexity of Algorithm 8.5 is $O(|\mathrm{var}_{X_u} M_{\mathrm{simp}}|^2)$. The next theorem ensures that no MSS set is lost in the simplification steps.

Theorem 8.4. $\mathrm{mss} M_{\mathrm{diff}} = \mathrm{mss} M_{\mathrm{simp}}$

Table 8.5 *The model M^{DLSM}_{over} that is input to Algorithm 8.5. The variables that are involved in the merging are denoted with bold crosses.*

model	unknown				known		
	f_{in}	f_{out}	h	f	u	y_h	y_f
e_1	**X**				X		
e_3	**X**	X	X				
e_4		X	X				
e_7		X		**X**			
e_{10}			X			X	
e_{14}				**X**			X

Table 8.6 *The model M^{DLSM}_{simp}.*

model	unknown		known		
	f_{out}	h	u	y_h	y_f
$\{e_1, e_3\}$	X	X	X		
$\{e_7, e_{14}\}$	X				X
$\{e_4\}$	X	X			
$\{e_{10}\}$		X		X	

The proof is at the end of this section.

Example 8.7 Consider Example 8.6. First we consider the DLSM in Table 8.3. The overdetermined part M^{DLSM}_{over} is shown in Table 8.5. The second simplification step searches for variables which belong only to two equations. The first time step (3) is applied to M^{DLSM}_{over}, it is found that f_{in} is included in $\{e_1, e_3\}$ and f in $\{e_{10}, e_{14}\}$. In Table 8.5 the variables that are involved in the merging are denoted with bold crosses. The structural model obtained after each of these two equation sets are merged is shown in Table 8.6. Since two merges were done, the algorithm searches for simplifications once more but now in the structural model in Table 8.6. In this model, both unknown variables are contained in 3 equations. Hence no more simplification can be done and the structural model in Table 8.6 is the result of

Table 8.7 *An analytical model corresponding to M^{DLSM}_{simp} in Table 8.6.*

model	expression
$\{e_1, e_3\}$	$\dot{h} = u - f_{out}$
$\{e_7, e_{14}\}$	$y_f = f_{out}$
$\{e_4\}$	$h = f^2_{out}$
$\{e_{10}\}$	$y_h = h$

Table 8.8 *The structural model* M^{DSSM}_{simp}.

model	unknown f_{out}	known u	y_h	y_f	$\dot{y}_f$
$\{e_7, e_{14}\}$	X			X	
$\{e_4, e_{10}\}$	X		X		
$\{e_1, e_3, \dot{e}_4, \dot{e}_7, \dot{e}_{14}\}$	X	X			X

Table 8.9 *An analytical model corresponding to* M^{DSSM}_{simp} *in Table 8.8*

model	expression
$\{e_7, e_{14}\}$	$y_f = f_{out}$
$\{e_4, e_{10}\}$	$y_h = f^2_{out}$
$\{e_1, e_3, \dot{e}_4, \dot{e}_7, \dot{e}_{14}\}$	$u = f_{out}(1 + 2\,\dot{y}_f)$

the simplification steps, i.e. M^{DLSM}_{simp}. An analytical interpretation of the simplified model is shown in Table 8.7.

Now we consider the DSSM, M^{DSSM}_{over}, below the lowest horizontal line in Table 8.4. The second simplification step searches for variables which belong only to two equations. In the first search for such variables, the algorithm finds f in $\{e_{14}, e_7\}$, h in $\{e_4, e_{10}\}$, $\dot{f}$ in $\{\dot{e}_{14}, \dot{e}_7\}$, and $\dot{f}_{out}$ in the models produced by $\{\dot{e}_{14}, \dot{e}_7\}$ and $\dot{e}_4$. The last merging makes a set $\{\dot{e}_4, \dot{e}_7, \dot{e}_{14}\}$. Continuing in the same way it is found that $\dot{h}$ is contained in $\{\dot{e}_4, \dot{e}_7, \dot{e}_{14}\}$ and e_3. Finally the algorithm finds f_{in} in $\{e_3, \dot{e}_4, \dot{e}_7, \dot{e}_{14}\}$ and e_1. Simplifications are made when step (3) is applied and therefore step (2) and (3) are performed once more. The second time no simplifications are made and the simplification step is therefore complete. The structural model M^{DSSM}_{simp} is shown in Table 8.8 and a corresponding analytical model is shown in Table 8.9.

Before we prove Theorem 8.4 a Lemma is presented.

Lemma 8.5. *Given that*

1. *a model* M *has the property* $\forall \bar{X} \neq \varnothing$, $\bar{X} \subseteq \mathrm{var}_{X_u} M : |\bar{X}| < |\mathrm{equ}_M(\bar{X})|$,
2. $X \neq \varnothing$,
3. $1 + |X| = |\mathrm{equ}_M(X)|$,
4. E *is an MSS set and* $E \subseteq M$,
5. $E \cap \mathrm{equ}_M(X) \neq \varnothing$,

then $\mathrm{equ}_M(X) \subseteq E$.

The proof of Lemma 8.5 follows immediately after the proof of Theorem 8.4 which is presented next.

Proof. From Lemma 3.5 it follows that for any $E \in \text{mss}M_{\mathtt{diff}}$ implies that $E \subseteq M^+_{\mathtt{diff}} = M_{\mathtt{over}}$. Hence $\text{mss}M_{\mathtt{diff}} = \text{mss}M_{\mathtt{over}}$. It remains to prove that $\text{mss}M_{\mathtt{over}} = \text{mss}M_{\mathtt{simp}}$. Since Algorithm 8.5 only changes the model $M_{\mathtt{simp}}$ in step 3 it is sufficient to prove that this operation on $M_{\mathtt{simp}}$ preserves the MSS sets included. Let the structural models $M_{\mathtt{simp}}$ before and after a simplification in step (3) be denoted M_1 and M_2 respectively.

Since the model $M_{\mathtt{over}}$ is structurally overdetermined, it satisfy (1) in Lemma 8.5. Assume that Algorithm 8.5 finds that $x \in \text{var}_{X_u} M_1$ fulfills $|\text{equ}_{M_1}(x)| = 2$. If $X = \{x\}$ in Lemma 8.5 then (2), and (3) in Lemma 8.5 are fulfilled. Take an arbitrary MSS set $E \subseteq M_1$ i.e. property (4). in Lemma 8.5. There are two cases to consider, either $E \cap \text{equ}_{M_1}(x) = \varnothing$ is true or $E \cap \text{equ}_{M_1}(x) \neq \varnothing$ is true.

1. If $E \cap \text{equ}_{M_1}(x) = \varnothing$, then E is not involved in the simplification and it is clear that $E \subseteq M_2$.
2. Otherwise it holds that $E \cap \text{equ}_{M_1}(x) \neq \varnothing$. This is condition (5). in Lemma 8.5. Since all 5 conditions in Lemma 8.5 are fulfilled the conclusion $\text{equ}_{M_1}(x) \subseteq E$ follows. This means that $\text{equ}_{M_1}(x)$ could be considered as one model derived from $\text{equ}_{M_1}(x)$ by eliminating the variable x. Hence $E \subseteq M_2$. Moreover if M_1 has property (1). in Lemma 8.5, then M_2 has also this property because
$$\begin{aligned}\forall \bar{X} \neq \varnothing, \bar{X} \subseteq \text{var}_{X_u} M_2 : |\bar{X}| = |\bar{X} \cup \{x\}| - 1 <\\ < |\text{equ}_{M_1}(\bar{X} \cup \{x\})| - 1 = |\text{equ}_{M_2}(\bar{X})|\end{aligned}$$

Since E was an arbitrary MSS set then $\text{mss}M_1 = \text{mss}M_2$. Algorithm 8.5 applies step (3) repeatedly, hence $\text{mss}M_{\mathtt{over}} = \text{mss}M_{\mathtt{simp}}$. □

Now we continue with the proof of Lemma 8.5.

Proof. Let $E' = E \cap \text{equ}_M(X)$ and $X_{E'} = \text{var}_X E'$. Since $E \cap \text{equ}_M(X) \neq \varnothing$, then $\exists e \in E' \exists x \in X : x \in \text{var}_X\{e\}$. It follows that $\varnothing \neq \{x\} \subseteq \text{var}_X E' = X_{E'}$. Suppose that $X \backslash X_{E'} = \varnothing$. Then $X_{E'} = X$ because $X_{E'} \subseteq X$. Apply Lemma 3.5 to $X \subseteq \text{var}_X E$ where $X \neq \varnothing$ then it follows that $|\text{equ}_E(X)| > |X|$. Then $|E'| > |X|$ since the definition of E' gives $|E'| = |\text{equ}_E(X)|$. Condition (3) implies an upper bound on $|E'|$,

$$|E'| = |E \cap \text{equ}_M(X)| \leq |\text{equ}_M(X)| = 1 + |X|. \tag{8.4}$$

From inequality (8.4) and $|E'| > |X|$ it follows that $\text{equ}_M(X) = E'$, hence $\text{equ}_M(X) \subseteq E$. Suppose contrary that $X \backslash X_{E'} \neq \varnothing$. Now, condition (1) of the system M where $\bar{X} = X \backslash X_{E'}$ gives the inequality

$$|X \backslash X_{E'}| \leq |\text{equ}_M((X \backslash X_{E'}))| - 1. \tag{8.5}$$

Consider the negation of the conclusion in Lemma 8.5, i.e. $\text{equ}_M(X)\backslash E' \neq \varnothing$, E is an MSS set, and $X_{E'} \neq \varnothing$. From Lemma 3.5 where $\bar{X} = X_{E'}$ it follows that

$$|X_{E'}| \leq |\text{equ}_E(X_{E'})| - 1. \tag{8.6}$$

Add inequality (8.5) and (8.6)

$$\begin{aligned} |X| = |X_{E'}| + |X\backslash X_{E'}| \leq \\ \leq |\text{equ}_E(X_{E'})| + |\text{equ}_M((X\backslash X_{E'}))| - 2 \leq \\ \leq |\text{equ}_M(X)| - 2. \end{aligned} \tag{8.7}$$

The last inequality in (8.7) follows since $\text{equ}_E(X_{E'}) \cap \text{equ}_M((X\backslash X_{E'})) = \varnothing$. Condition 3. implies a contradiction $|X| + 2 \leq |\text{equ}_M(X)| = |X| + 1$. Hence, $\text{equ}_M(X) \subseteq E$. □

8.5 Finding MSS Sets

After the simplification steps are completed, step (e) in Algorithm 8.1 finds all MSS sets in the simplified model M_{simp}. This section explains how the MSS sets are found. The task is to find all MSS sets in the model M_{simp} with equations $\{e_1, \cdots, e_n\}$. Let $M^k = \{e_k, \cdots, e_n\}$ be the last $n - k + 1$ equations. Let E be the current set of equations that is examined. The set of MSS sets found is denoted γ_{mss}. Then the following algorithm finds all MSS sets in M_{simp}.

Algorithm 8.6.
Input: The model M_{simp}.

1. *Set* $k = 1$ *and* $\gamma_{mss} = \varnothing$.
2. *Choose equation* e_k. *Let* $E = \{e_k\}$ *and* $X = \varnothing$.
3. *Find all MSS sets that are subsets of* M^k *and include equation* e_k.
   a) *Let* $\tilde{X} = \text{var}_{X_u}(E)\backslash X$ *be the unmatched variables.*
   b) *If* $\tilde{X} = \varnothing$, *then* E *is an MSS set. Insert* E *into* γ_{mss}.
   c) *Else take a remaining variable* $\tilde{x} \in \tilde{X}$ *and let* $X = X \cup \{\tilde{x}\}$. *Let* $\tilde{E} = \text{equ}_{M^k\backslash E}(\tilde{x})$ *be the remaining equations. For all equations* e *in* $\tilde{E}$ *let* $E = E \cup \{e\}$ *and goto step (a).*
4. *If* $k < n$ *set* $k = k + 1$ *and goto step (2).*

Output: The set of MSS sets found, i.e. γ_{mss}.

Algorithm 8.6 finds all MSS sets in M_{simp} according to the next theorem.

Theorem 8.6. $\gamma_{mss} = \text{mss}M_{simp}$

Table 8.10 *The DLSM obtained after the simplification step. To simplify the notation, the sets of equations are renamed according to the left column.*

renamed model	model	unknown		known		
		f_{out}	h	u	y_h	y_f
1	$\{e_1, e_3\}$	X	X	X		
2	$\{e_7, e_{14}\}$	X				X
3	$\{e_4\}$	X	X			
4	$\{e_{10}\}$		X		X	

In (Pulido & Alonso 2002) there is a similar algorithm for computing *minimal evaluation chain* (MEC) which is almost the same as the MSS sets. The difference is that a MEC has to include known variables. The next example illustrates Algorithm 8.6.

Example 8.8 Consider the structural model in Table 8.6. To simplify the notation, let the equations be renamed as defined in Table 8.10. The important variables computed when Algorithm 8.6 is applied to the structural model in Table 8.10 are shown in Table 8.11. First in step (1) k and γ_{mss} are initialized. Then in step (2) the set of current equation is $E = \{1\}$ where 1 is the name of the first equation and the set of matched variables $X = \varnothing$. These two variables are shown in the first row and in the second and third column respectively in Table 8.11. In step (3a) the unmatched unknown variables in E are stored. In this case equation 1 includes the unknown variables f_{out} and h, i.e. $\tilde{X} = \{f_{out}, h\}$ as shown in the fourth column in Table 8.11. Since $\tilde{X} \neq \varnothing$ step (3c) is done. First let $\tilde{x} := f_{out} \in \tilde{X}$ and set $X = X \cup \{\tilde{x}\} = \{f_{out}\}$. In Table 8.10 it is seen that the equations $\{1, 2, 3\}$ contain f_{out}. The algorithm finds that equations $\{2, 3\}$ in the set $M^1 \backslash E = \{1, 2, 3, 4\} \backslash \{1\}$ include f_{out}. Hence $\tilde{E} = \{2, 3\}$ in step (3c). The first equation in $\tilde{E} = \{2, 3\}$ is added to E, i.e. $E = \{1, 2\}$ and $X = \{f_{out}\}$. This is seen in the second row of Table 8.11. Now continuing in the same way as described the computed variables can be seen Table 8.11. In the third row in Table 8.11 it is interesting to note that $\tilde{X} = \varnothing$. Then in step (3b) the set E is inserted into γ_{mss} because it is an MSS set. The MSS set obtained is $\{1, 2, 3\}$ and it has the following structure

renamed model	unknown		known		
	f_{out}	h	u	y_h	y_f
1	X	X	X		
2	**X**				X
3	X	**X**			

(8.8)

This MSS was found because first equation 1 was chosen and then there is a perfect matching of all unknown variables of E into $E \backslash \{1\}$ as can be seen in (8.8). This matching defines a 2-spanning tree in the following way. First equation 1 is the root of the tree. The children of the root are its unknown variables, i.e. f_{out} and

h. The matching defines the child of each variable vertex, in this example 2 is a child of f_{out} and 3 is a child of h. If the equations in E are rearranged in the same order as they are added to E in Algorithm 8.6, then the parent vertex to each variable vertex is defined as the first equation that contains the variable. In this way a spanning tree is built for any MSS set.

Before the row that starts with a 2 in Table 8.11 the algorithm has found all MSS sets that include equation 1, i.e. $\{1,2,3\}$, $\{1,2,4\}$, and $\{1,3,4\}$. Since all MSS sets are found that include equation 1, it remains to find MSS sets in $M^2 = \{2,3,4\}$. Continuing in this way the MSS sets found are $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$. In the original notation the MSS sets are

$$\gamma_{mss} = \{\{e_1, e_3, e_4, e_7, e_{14}\}, \{e_1, e_3, e_7, e_{10}, e_{14}\}, \{e_1, e_3, e_4, e_{10}\}, \{e_4, e_7, e_{10}, e_{14}\}\} \tag{8.9}$$

Finally we also present the MSS sets found in the DSSM. The 3 MSS sets contained in the model shown in Table 8.8 are

$$\gamma_{mss} = \{\{e_4, e_7, e_{10}, e_{14}\}, \{e_1, e_3, \dot{e}_4, e_7, \dot{e}_7, e_{14}, \dot{e}_{14}\}, \{e_1, e_3, e_4, \dot{e}_4, \dot{e}_7, e_{10}, \dot{e}_{14}\}\} \tag{8.10}$$

Note that $\{e_4, e_7, e_{10}, e_{14}\}$ is found both in (8.9) and (8.10). This follows from the fact that all these equations are static equations and it is only with respect to differentiation these two methods are different. Furthermore we see that $\{e_1, e_3, \dot{e}_4, e_7, \dot{e}_7, e_{14}, \dot{e}_{14}\}$ in (8.10) uses the same original equations as $\{e_1, e_3, e_4, e_7, e_{14}\}$ in (8.9). There are no MSS sets in (8.10) that use exactly the same original equations as the second and third MSS in (8.9), because the differentiation limitations $m(z)$ are too low to find such sets. In (8.10) another MSS set is found that uses the original equations $\{1, 3, 4, 7, 10, 14\}$ and fulfills the differentiation limitations.

Next we prove Theorem 8.6.

Proof. To show the inclusion $\gamma_{mss} \subseteq \text{mss}M_{simp}$, we will take an arbitrary $E \in \gamma_{mss}$ and show that E is an MSS set. The set E is an MSS set if and only if $\forall e \in E$ there exist a perfect matching in $\mathcal{G}(E\backslash\{e\}, \text{var}_{X_u} E)$ according to Lemma 3.4. The goal is to find a perfect matching in $\mathcal{G}(E\backslash\{e\}, \text{var}_{X_u} E)$ for all e.

Number the equations in E as they were found in Algorithm 8.6, i.e. $E = \{e_1, \ldots, e_n\}$. Let $E_j = \{e_1, \ldots, e_j\}$ be the first j equations found. Note that when Algorithm 8.6 stores E in step (3b) it holds that $j = n$. If $n = 1$, then $\text{var}_{X_u} E_1 = \tilde{X}_1 = \varnothing$ in (3b) and $E_1 = \{e_1\}$ is an MSS set. Otherwise, if $n \geq 2$ take an arbitrary $e_a \in E$. The next paragraph shows that there will be a perfect matching in $\mathcal{G}(E\backslash\{e_a\}, \text{var}_X E)$ for all $a \in \{1, 2, 3, \ldots, n\}$.

If $n \geq 2$ the algorithm finds a complete matching of $\text{var}_{X_u} E$ into E. This assignment is $\{e_2, x_1\}$, $\{e_3, x_2\}$, ... ,$\{e_n, x_{n-1}\}$. If $a = 1$ a perfect matching is the previous assignment. If $a \neq 1$, then all variables but x_{a-1} have an assignment. The next paragraph shows that it is always possible to construct an augmenting path between e_a and e_1. This path defines a reassignment such that the new assignment is a perfect matching in $\mathcal{G}(E\backslash\{e_a\}, \text{var}_{X_u} E)$.

Algorithm 8.6 picks an equation e_i in step (3c) only if x_{i-1} is included in e_i. In step (3c) a search for x_{i-1} is performed only if $x_{i-1} \in \text{var}_{X_u} E_{i-1}$ according to

Table 8.11 *The time evolution of the important variables in Algorithm 8.6 applied to the structural model in Table 8.10.*

e_k	E	X	$\tilde{X} = \mathrm{var}_{X_u}(E)\backslash X$		MSS
1	$\{1\}$	$\varnothing$	$\{f_{out}, h\}$		
	$\{1,2\}$	$\{f_{out}\}$	$\{h\}$		
	$\{1,2,3\}$	$\{f_{out}, h\}$	$\varnothing$	$\Rightarrow$	$\{1,2,3\}$
	$\{1,2,4\}$	$\{f_{out}, h\}$	$\varnothing$	$\Rightarrow$	$\{1,2,4\}$
	$\{1,3\}$	$\{f_{out}\}$	$\{h\}$		
	$\{1,3,4\}$	$\{f_{out}, h\}$	$\varnothing$	$\Rightarrow$	$\{1,3,4\}$
2	$\{2\}$	$\varnothing$	$\{f_{out}\}$		
	$\{2,3\}$	$\{f_{out}\}$	$\{h\}$		
	$\{2,3,4\}$	$\{f_{out}, h\}$	$\varnothing$	$\Rightarrow$	$\{2,3,4\}$
3	$\{3\}$	$\varnothing$	$\{f_{out}, \{h\}$		
4	$\{4\}$	$\varnothing$	$\{h\}$		

Table 8.12 *Test quantities for the MSS sets in (8.9).*

$H_i^0 : sys \in \Phi_i$	test quantity T_i
$\phi(p = NF \wedge t = NF \wedge p_1 = NF \wedge s_2 = NF)$	$T_1(s) = \frac{1}{s+1}(s\,(y_f^2) - u + y_f)$
$\phi(p = NF \wedge p_1 = NF \wedge s_1 = NF \wedge s_2 = NF)$	$T_2(s) = \frac{1}{s+1}(s\,y_h - u + y_f)$
$\phi(p = NF \wedge t = NF \wedge s_1 = NF)$	$T_3(s) = \frac{1}{s+1}(s\,y_h - u + \sqrt{y_h})$
$\phi(t = NF \wedge p_1 = NF \wedge s_1 = NF \wedge s_2 = NF)$	$T_4 = y_h - y_f^2$

step (3a). The conclusion is that for any $i \in \{2, 3, \ldots, n\}$, it is possible to find an equation e_b such that $x_{i-1} \in \mathrm{var}_{X_u} e_b$ and $b < i$. This is a sufficient condition to find an augmenting path from e_a to e_1.

Starting in equation e_a the assignment imply the first edge to x_{j-1}. From the previous paragraph there is an edge between e_{b_1} and x_{j-1} where $b_1 < j$. This can be repeated until $b_k = 1$. Since b_1 is finite and (b_i) is a strictly decreasing list of natural numbers, it follows that k is finite. Reassign the equations and variables included in the augmenting path so $\{e_{b_1}, x_{a-1}\}$, $\{e_{b_2}, x_{b_1-1}\}$, $\{e_{b_3}, x_{b_2-1}\}$, ... ,$\{e_1, x_{b_{k-1}-1}\}$. This assignment is a perfect matching in $\mathcal{G}(E\backslash\{e_a\}, \mathrm{var}_{X_u} E)$. Using Theorem 3.4 the conclusion is that $\gamma_{mss} \subseteq \mathrm{mss}M_{simp}$.

The second part of the proof shows that $\mathrm{mss}M_{simp} \subseteq \gamma_{mss}$. Take an arbitrary $E \in \mathrm{mss}M_{simp}$. Let e_1 be the first equation in E that Algorithm 8.6 picks in step (2). If $E = \{e_1\}$ then $\mathrm{var}_{X_u} e_1 = \varnothing$ and the algorithm finds the MSS set immediately in step (3b). If $\{e_1\} \subset E$ then according to Theorem 3.4 there is a perfect matching in $\mathcal{G}(E\backslash\{e_1\}, \mathrm{var}_X E)$. Take any perfect matching in $\mathcal{G}(E\backslash\{e_1\}, \mathrm{var}_X E)$.

This perfect matching is $\{x_1, e_2\}$, $\{x_2, e_3\} \ldots \{x_{|E|-1}, e_{|E|}\}$. We will now show that the algorithm will find this perfect matching. The enumeration of the variables are defined step by step as they are found in Algorithm 8.6.

Since $\{e_1\}$ is not an MSS set then $\mathrm{var}_X e_1 \neq \varnothing$. The algorithm picks a $x \in \mathrm{var}_X e_1$ in step (3c). This x is defined as x_1 by the algorithm. The given perfect matching assigns x_1 to e_2. This is only possible if $e_2 \in \mathrm{equ}_{M^k \setminus \{e_1\}}(x_1)$. Then $\tilde{E}_1 = \mathrm{equ}_{M^k \setminus \{e_1\}}(x_1)$ in step (3c). Finally step (3c) will assign x_1 once at a time to all $e \in \mathrm{equ}_{M^k \setminus \{e_1\}}(x_1)$. Particularly the algorithm will assign x_1 to e_2.

Now, suppose that the algorithm has assigned $\{x_1, e_2\}$, $\{x_2, e_3\}$, ... ,$\{x_i, e_{i+1}\}$ for any $1 \leq i \leq |E| - 2$. This means that step (3c) is just done and the algorithm will start in step (3a) again.

The current value of the variables are

$$\begin{aligned} E_{i+1} &= \{e_1, e_2, \ldots, e_{i+1}\} \\ X_{i+1} &= \{x_1, x_2, \ldots, x_i\}. \end{aligned}$$

In step (3a) $\tilde{X}_{i+1} = \mathrm{var}_{X_u} E_{i+1} \setminus X_{i+1}$. From the assumption it follows that E_{i+1} is not structurally singular, because $i \leq |E| - 2$, hence $\mathrm{var}_{X_u} E_{i+1} \setminus X_{i+1} \neq \varnothing$. This implies that $\tilde{X}_{i+1} \neq \varnothing$. Hence it must be at least one variable in $\tilde{X}_{i+1}$. The variable that the algorithm picks is denoted x_{i+1}.

The variable x_{i+1} is assigned e_{i+2} according to the given matching. Then $e_{i+2} \in \mathrm{equ}_{M^k}(x_{i+1})$ and especially $e_{i+2} \in \mathrm{equ}_{(M^k \setminus E_{i+1})}(x_{i+1})$. Hence $e_{i+2} \in \tilde{E}_{i+1}$ in step (3c). Since step (3c) assign x_{i+1} to all $e \in \tilde{E}_{i+1}$ one at a time, the algorithm will particularly assign x_{i+1} to e_{i+2}.

Now, $E_{|E|} = \{e_1, \ldots, e_n\}$, $X_{|E|} = \{x_1, \ldots, x_{|E|-1}\}$, and Algorithm 8.6 starts at step (3a). Since E is an MSS set it follows that $|E| > |\mathrm{var}_{X_u} E|$. From the definition of $\{x_1, \ldots, x_{|E|-1}\}$, it follows that $\mathrm{var}_{X_u} E = \{x_1, \ldots, x_{|E|-1}\}$. Then $\tilde{X}_{|E|} = \mathrm{var}_{X_u} E \setminus \{x_1, \ldots, x_{|E|-1}\} = \varnothing$. This is detected in step (3b) and the algorithm conclude that $E_{|E|} = E$ is an MSS set. Hence $\mathrm{mss} M_{simp} \subseteq \gamma_{mss}$.

□

8.6 Evaluating Isolability and Extracting Behavioral Models

Let, as before, $\mathcal{I}_d$ be the *desired isolability.* More information about isolability can be found in Chapter 4 and desired isolability is defined in Section 4.7. The relation $\mathcal{I}_d$ can be specified by the demands of the isolability of the diagnostic system or just set to full isolability.

Before we describe step (g) and (h) in Algorithm 8.2 a desired isolability is defined for the water-tank example. An isolability analysis is used to guide the choice of desired isolability.

Assume that we want to design a diagnostic system that can detect and isolate all single faults. Let $\mathcal{B}_{01}$ be the set of system behavioral modes that either are single faults or **NF**. Then the desired isolability can be defined as

$$\mathcal{I}_d = \{(b_i, b_j) | b_i \in \mathcal{B}_{01}, b_j \in \mathcal{B}_{01}, b_i \neq b_j\} \tag{8.11}$$

To find out if this isolability can be obtained, a structural isolability analysis of the diagnostic model described for example as a DLSM can be done. The structural isolability $\mathcal{I}_{sp}^{\mathbb{M}}(\langle M_{b_i}\rangle)$ introduced in (6.52) is

present mode	necessary interpreted mode							
	NF	$\mathbf{PS}_p$	$\mathbf{UF}_p$	$\mathbf{C}_t$	$\mathbf{L}_{p_1}$	$\mathbf{UF}_{s_1}$	$\mathbf{L}_{p_2}$	$\mathbf{UF}_{s_2}$
NF	X		X	X	X	X	X	X
$\mathbf{PS}_p$		X	X					
$\mathbf{UF}_p$			X					
$\mathbf{C}_t$				X				
$\mathbf{L}_{p_1}$					X			
$\mathbf{UF}_{s_1}$						X		
$\mathbf{L}_{p_2}$							X	
$\mathbf{UF}_{s_2}$								X

(8.12)

If the structurally overdetermined models are found the isolability $\mathcal{I}_{sp}^{\mathbb{M}}(\langle M_{b_i}^{+}\rangle)$ is

present mode	necessary interpreted mode							
	NF	$\mathbf{PS}_p$	$\mathbf{UF}_p$	$\mathbf{C}_t$	$\mathbf{L}_{p_1}$	$\mathbf{UF}_{s_1}$	$\mathbf{L}_{p_2}$	$\mathbf{UF}_{s_2}$
NF	X		X	X	X	X	X	X
$\mathbf{PS}_p$		X	X					
$\mathbf{UF}_p$			X					
$\mathbf{C}_t$				X				
$\mathbf{L}_{p_1}$					X			X
$\mathbf{UF}_{s_1}$						X		
$\mathbf{L}_{p_2}$	X		X	X	X	X	X	X
$\mathbf{UF}_{s_2}$								X

(8.13)

Note that the process of making the assumptions weaker, as described in Section 4.5, was done in order to obtain (8.12) and (8.13). This means that the assumptions for e_5, e_6, e_8, e_9, e_{12}, and e_{13} are set to $\mathcal{B}$.

It can be realized that the set of all MSS sets have the same structural isolability as (8.13). Since the best structural isolability of any set of MSS sets is (8.13), the desired isolability can be chosen as (8.13) instead of (8.11). By changing the desired isolability fewer behavioral models are analyzed in Algorithm 8.1 without loosing any structural isolability.

The structural isolability $\mathcal{I}_s^{\mathbb{M}}(\gamma)$ where $\gamma = \text{mss}M_{\mathbf{NF}}^{DLSM}$ computed in step (g) in Algorithm 8.2 is

present mode	necessary interpreted mode							
	NF	$\mathbf{PS}_p$	$\mathbf{UF}_p$	$\mathbf{C}_t$	$\mathbf{L}_{p_1}$	$\mathbf{UF}_{s_1}$	$\mathbf{L}_{p_2}$	$\mathbf{UF}_{s_2}$
NF	X	**X**	X	X	X	X	X	X
$\mathbf{PS}_p$		X	X					
$\mathbf{UF}_p$		**X**	X					
$\mathbf{C}_t$				X				
$\mathbf{L}_{p_1}$					X			X
$\mathbf{UF}_{s_1}$						X		
$\mathbf{L}_{p_2}$	X	**X**	X	X	X	X	X	X
$\mathbf{UF}_{s_2}$					**X**			X

(8.14)

The crosses marked in bold denote missing isolability properties in order to fulfill the desired isolability in (8.13).

Now, step (h) in Algorithm 8.2 is described. Suppose that the $(b_i, b_j) \in \mathcal{I}_d$ but $(b_i, b_j) \notin \mathcal{I}_s^{\mathbb{M}}(\gamma)$ where γ is the set of MSS sets found. This means that the desired isolability is not fulfilled with the MSS sets found in the behavioral models analyzed so far. However, it could still be possible to structurally isolate behavioral mode b_i from behavioral mode b_j by search for MSS sets in behavioral model M_{b_j}. If the set of behavioral modes $b \in B \subseteq \mathcal{B}$ have been applied to step (a)-(e) Algorithm 8.1, step (a) in Algorithm 8.1 is applied to a $b_j \in B$ if

$$\exists b_i \in \mathcal{B} : (b_i, b_j) \in \mathcal{I}_d \backslash \mathcal{I}_s^{\mathbb{M}}(\gamma) \tag{8.15}$$

Then if there is an MSS set $M \subseteq M_{b_j}$ such that $b_i \notin \operatorname{ass} M$ then the desired structural isolability property is obtained. If no b_j exists such that (8.15) is fulfilled then go to step (i) in Algorithm 8.2.

Now we continue with the water-tank example. The first column in (8.14) where missing isolability properties appear is in column $\mathbf{PS}_p$. Step (a) in Algorithm 8.1 is applied to $\mathbf{PS}_p$ according to (8.14). The additional MSS sets found in the model $M_{\mathbf{PS}_p}^{DLSM} = \{e_2, e_3, e_4, e_7, e_{10}, e_{11}, e_{14}\}$ are

$$\{\{e_2, e_3, e_4, e_7, e_{14}\}, \{e_2, e_3, e_7, e_{10}, e_{14}\}, \{e_2, e_3, e_4, e_{10}\}\} \tag{8.16}$$

Using these 3 additional MSS sets the structural isolability is

present mode	necessary interpreted mode							
	NF	$\mathbf{PS}_p$	$\mathbf{UF}_p$	$\mathbf{C}_t$	$\mathbf{L}_{p_1}$	$\mathbf{UF}_{s_1}$	$\mathbf{L}_{p_2}$	$\mathbf{UF}_{s_2}$
NF	X		X	X	X	X	X	X
$\mathbf{PS}_p$		X	X					
$\mathbf{UF}_p$			X					
$\mathbf{C}_t$				X				
$\mathbf{L}_{p_1}$					X			X
$\mathbf{UF}_{s_1}$						X		
$\mathbf{L}_{p_2}$	X		X	X	X	X	X	X
$\mathbf{UF}_{s_2}$					**X**			X

(8.17)

Note that the missing isolability properties in the column corresponding to $\mathbf{PS}_p$ are with the additional MSS sets obtained. Now the only column with a missing isolability property is the column corresponding to $\mathbf{L}_{p_1}$. The additional MSS sets found in $M^{DLSM}_{\mathbf{L}_{p_1}}$ are

$$\{\{e_1, e_3, e_4, e_8, e_9, e_{14}\}, \{e_1, e_3, e_8, e_9, e_{10}, e_{14}\}, \{e_4, e_8, e_9, e_{10}, e_{14}\}\} \tag{8.18}$$

With these additional MSS sets the desired isolability in (8.13) is obtained.

8.7 Selecting a Small Subset of MSS Sets

It is not unusual that the number of MSS sets or models γ' found is large. Many of the models probably use almost as many equations as unknown variables in the entire system. These models usually rely on too many uncertainties to be useable for fault isolation. Small models are more robust and are usually sensitive to fewer faults. Therefore a goal can be to find a set of models with the highest possible *robustness* and with the same structural isolability as the set of all models.

Assume that it is possible to calculate a real number for each model that is inversely proportional to the robustness of the model. Let this number for model m be denoted n_m. This number can for example be as in this thesis, the number of equations in each model.

Now, let $\Psi(\gamma, i)$ be a function that has a set γ of models and an isolability property i as arguments and returns the models of γ with property i.

Let γ' be the set of models that is the input to step (i) in Algorithm 8.2. Furthermore $\gamma \subseteq \gamma'$ is a subsets of models with the maximum structural isolability and maximum robustness according to the defined values n_m if γ fulfills

$$\begin{aligned} &\forall i \in \mathcal{I}_s^{\mathbb{M}}(\gamma')\big(\Psi(\gamma', i) \neq \varnothing \rightarrow \\ &\rightarrow \min\nolimits_{\psi \in \Psi(\gamma', i)} n_\psi = \min\nolimits_{\psi \in \Psi(\gamma, i)} n_\psi\big) \end{aligned} \tag{8.19}$$

Note that γ is not unique, but the minimum number for each $i \in \mathcal{I}_s^{\mathbb{M}}(\gamma')$ is unique.

Step (i) in Algorithm 8.2 start to sort the models in γ' in an increasing order of robustness. The models are examined in the rearranged order. If a model increases the isolability, then the model is selected. This means that for each isolability property, the smallest models with this isolability property will be one of the chosen models. In this way the final output from Algorithm 8.2 will be a small set γ of models with highest possible isolability and highest possible robustness, i.e. γ fulfills expression (8.19). An upper bound to $|\gamma|$ is $|\mathcal{I}_d|$.

For the water-tank example Step (i) in Algorithm 8.2 rearranges the models from (8.9), (8.16), and (8.18) with increasing size as shown in Table 8.13. It is possible to define an upper limit of the isolability requirement fulfillment as $|\mathcal{I}_s^{\mathbb{M}}(\gamma)|/|\mathcal{I}_d|$ where γ is the set of selected models. This number is for the water-tank example shown in the right column of Table 8.13 when the MSS sets are added one at a time. The 5:th, 7:th, 9:th, and 10:th MSS set in Table 8.13 does not increase the isolability requirement fulfillment and is therefore rejected. Hence the MSS sets 1,

Table 8.13 *The accumulated isolability requirement fulfillment for MSS sets found in DLSM of the water-tank example.*

	MSS set	accumulated isolability fulfillment
1	{1, 3, 4, 10}	38%
2	{2, 3, 4, 10}	55%
3	{4, 7, 10, 14}	74%
4	{1, 3, 4, 7, 14}	86%
5	{2, 3, 4, 7, 14}	86%
6	{1, 3, 7, 10, 14}	98%
7	{2, 3, 7, 10, 14}	98%
8	{4, 8, 9, 10, 14}	100%
9	{1, 3, 4, 8, 9, 14}	100%
10	{1, 3, 8, 9, 10, 14}	100%

(8.20)

Table 8.14 *Test quantities for the selected MSS sets in Table 8.13. Using physical properties of the known variables it is assumed that $u, y_h, y_f \geq 0$. Note that non-linearly transformed variables can be seen as input signals to the linear filters.*

$H_i^0 : sys \in \Phi_i$	Test quantity T_i
$\phi(p = NF \wedge t = NF \wedge s_1 = NF)$	$T_1 = \frac{1}{s+1}(s\, y_h - u + \sqrt{y_h})$
$\phi(p = PS \wedge t = NF \wedge s_1 = NF)$	$T_2 = \frac{1}{s+1}(s\, y_h + \sqrt{y_h})$
$\phi(t = NF \wedge p_1 = NF \wedge s_1 = NF \wedge s_2 = NF)$	$T_3 = y_h - y_f^2$
$\phi(p = NF \wedge t = NF \wedge p_1 = NF \wedge s_2 = NF)$	$T_4 = \frac{1}{s+1}(s\,(y_f^2) - u + y_f)$
$\phi(p = NF \wedge p_1 = NF \wedge s_1 = NF \wedge s_2 = NF)$	$T_5 = \frac{1}{s+1}(s\, y_h - u + y_f)$
$\phi(t = NF \wedge s_1 = NF \wedge s_2 = NF)$	$T_6 = \frac{s}{s+1}(y_h^{-1/2} y_f)$ if $y_h \neq 0$

2, 3, 4, 6, and 8 are selected. Examples of test quantities for these MSS sets can be seen in Table 8.14. Using the physical properties of the known variables it is assumed that $u, y_h, y_f \geq 0$.

8.8 Algorithm Improvements

If all MSS sets are to be found in a diagnostic model $\mathbb{M}$ with many behavioral modes, many behavioral models have to be analyzed. Since the finding of MSS sets is computationally complex it would decrease the amount of computations dramatically if only few behavioral models have to be analyzed. Two improvements to reduce the number of analyzed behavioral models are presented.

In Section 8.2 it was mentioned that behavioral models that are subsets of analyzed behavioral models need not be analyzed. This fact can be used to start with behavioral models that are maximal, i.e. no strict superset is a behavioral model.

The second improvement can be done if a DLSM is used. The idea is to use the merging technique in Algorithm 8.5 of the internal variables of each component-

Table 8.15 *A structural model containing the sufficient structural information to directly find all MSS sets of the diagnostic model in Table 2.2. The notation* (X) *in the first row denotes that* X *is included in* e_1 *but not in* e_2.

models	unknown					known		
	f_{in}	f_{out}	h	f	f_{int}	u	y_h	y_f
$e_1 \vee e_2$	X					(X)		
e_3	X	X	X					
$e_4 \vee \{e_5, e_6\}$		X	X					
$e_7 \vee \{e_8, e_9\}$		X		X				
e_{10}			X				X	
$e_{11} \vee \{e_{12}, e_{13}\}$				X	X			
e_{14}				X				X

behavioral mode. As an example, consider the tank in the water-tank example. The only internal variable is A. After merging equation e_5 and e_6 the resulting structural model of $\{e_5, e_6\}$ is the same as e_4. Generalizing the notation of structural models, the result of the first merging-step of component behavioral modes is shown in the column denoted models in Table 8.15. This model contains sufficient information to find all MSS sets that are contained in the 96 behavioral models of the diagnostic model in Table 2.2. Applying Algorithm 8.1 to the model in Table 8.15, which is structurally the same as $M_{\mathbf{NF}}^{DLSM}$ shown in Table 8.2, all MSS sets are found directly. All 20 MSS sets are shown in Table 8.16.

Table 8.16 *All 20 MSS sets of the diagnostic model of the water-tank system.*

	MSS set
1	$\{e_1, e_3, e_4, e_{10}\}$
2	$\{e_2, e_3, e_4, e_{10}\}$
3	$\{e_4, e_7, e_{10}, e_{14}\}$
4	$\{e_1, e_3, e_4, e_7, e_{14}\}$
5	$\{e_2, e_3, e_4, e_7, e_{14}\}$
6	$\{e_1, e_3, e_7, e_{10}, e_{14}\}$
7	$\{e_2, e_3, e_7, e_{10}, e_{14}\}$
8	$\{e_1, e_3, e_5, e_6, e_{10}\}$
9	$\{e_2, e_3, e_5, e_6, e_{10}\}$
10	$\{e_5, e_6, e_7, e_{10}, e_{14}\}$
11	$\{e_4, e_8, e_9, e_{10}, e_{14}\}$
12	$\{e_1, e_3, e_5, e_6, e_7, e_{14}\}$
13	$\{e_2, e_3, e_5, e_6, e_7, e_{14}\}$
14	$\{e_1, e_3, e_4, e_8, e_9, e_{14}\}$
15	$\{e_2, e_3, e_4, e_8, e_9, e_{14}\}$
16	$\{e_1, e_3, e_8, e_9, e_{10}, e_{14}\}$
17	$\{e_2, e_3, e_8, e_9, e_{10}, e_{14}\}$
18	$\{e_5, e_6, e_8, e_9, e_{10}, e_{14}\}$
19	$\{e_1, e_3, e_5, e_6, e_8, e_9, e_{14}\}$
20	$\{e_2, e_3, e_5, e_6, e_8, e_9, e_{14}\}$

9

Industrial Example: A Part of a Paper Mill

In this chapter Algorithm 8.1 and Algorithm 8.2 are applied to an industrial example. The maximum single fault isolability is calculated and a subset of MSS sets are selected which contains this isolability. The example is a stock preparation and broke treatment system of a paper mill located in Australia. The process is used for mixing and purifying recycled paper for production of new paper. An overview of the process is shown in Figure 9.1.

9.1 Process Description

After the preparation step the purified paper mixture is transferred to the *screen*. In the screen it is important that the mixture has a correct concentration of paper fibers and does not exceed a critical pressure. The process starts with recycled paper and water. The recycled paper has a high concentration of paper fibers. The two fluids are mixed in the *pulper* tank to a correct concentration. Looking in the right part of Figure 9.1, the *cyclone* purifies the paper mixture. This is done by spinning the fluid in the cyclone. The result is that large particles are collected at the bottom of the cyclone and clean paper mixture is collected at the top. A drawback to this method is that the purified mixture obtains a high pressure. To limit the outflow pressure from this part of the process, there is a pipe going back to a tank. When this pipe opens the pressure in the outflow mixture decreases. The return of the fluid increases the concentration of fibers in the tank. Therefore the mixture is diluted with water before entering the cyclone. For a more detailed description, see (Biteus 2001).

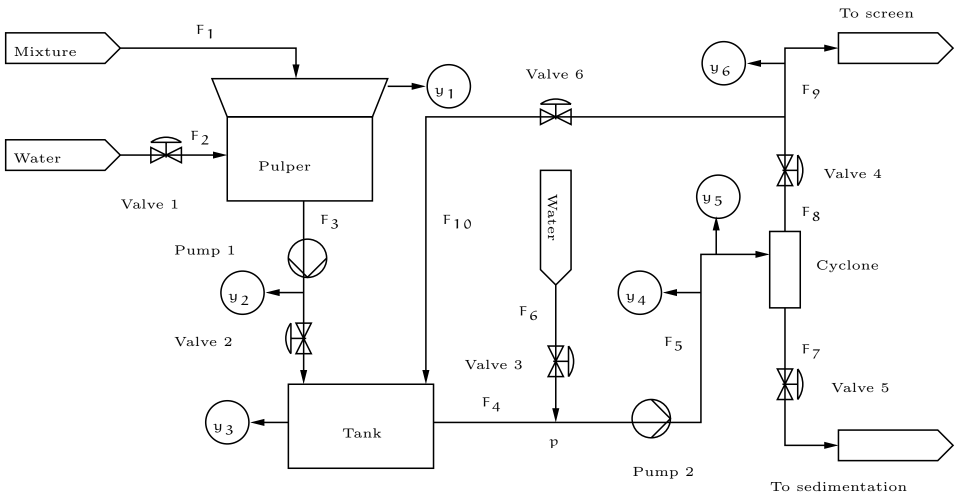

Figure 9.1 *A stock preparation and broke treatment system of a paper mill.*

9.2 Model Description

Most parts of the model are non-linear. It is only the tank and the pulper that are considered to be dynamic. The model has 4 states: the volumes x_1 and x_3 and the concentrations x_2 and x_4 in the pulper and in the tank respectively. There are 6 sensors in the process. Sensor y_1 and y_3 measure the water levels of the pulper and the tank respectively, y_2 and y_4 measure concentration, y_5 and y_6 measure pressure. The flows into and out of this process are known, i.e F_1, F_2, F_6, F_7, and F_9 are known. Moreover the concentrations of the fluids flowing into the process are constant and known, i.e. c_1, c_2, and c_6 are known. There are 6 valves with control signals u_i, where $i \in \{1,2,3,4,5,6\}$ and two pumps which have actuator signals z_{p1} and z_{p2}.

There are 21 single-faults that are analyzed. All sensors can have a constant offset fault f_{yi}, $i \in \{1,2,3,4,5,6\}$. All valves can have a constant offset in the actuator signal f_{ui}, $i \in \{1,2,3,4,5,6\}$. Clogging can occur in the pipes near the valves f_{ci}, $i \in \{1,2,3,4,5,6\}$ and also directly after the tank f_{c6}. Finally, the pumps can have a constant offset on the actuator signal f_{p1} and f_{p2}.

The model is described by 29 basic equations from which equations valid for different behavioral modes are deduced. The basic model equations are shown in Table 9.1. Equations $e_1, \ldots, e_4$ describe the dynamics; $e_5, \ldots, e_{14}$ are pressure loops; e_{15} relates the concentration in the junction after the tank with the flows F_4 and F_6; e_{16} and e_{17} describe the two pumps; $e_{18}, \ldots, e_{23}$ are valve equations; e_{24}, e_{25}, e_{26} are flow equations, and finally e_{27}, e_{28}, e_{29} are sensor equations for

Table 9.1 *The basic model equations for the process in Figure 9.1.*

e_1	$\dot{x}_1 - e_1(F_1 + F_2 - F_3) = 0$
e_2	$\dot{x}_2 - \frac{e_1(F_1(c_1+f_{y2}-y_2)+F_2(c_2+f_{y2}-y_2))}{y_1-f_{y1}} = 0$
e_3	$\dot{x}_3 - e_2(F_3 + F_{10} - F_4) = 0$
e_4	$\dot{x}_4 + \frac{e_2(F_{10}(f_{y4}+x_4-y_4)+F_3(f_{y2}+x_4-y_2))}{y_3-f_{y3}} = 0$
e_5	$g_1 - a_1 F_1^2 = 0$
e_6	$g_3 - F_2^2(a_2 + a_3 + b_1(f_{c1} + z_{u1})) = 0$
e_7	$k_1(y_1 - f_{y1}) + d_{11} - 1 + f_{p1}^2 z_{p1} - F_3^2(bcv3 + a_4 + a_5$
	$+a_6 + a_7 + b_2 f_{c2} + b_2 z_{u2}) = 0$
e_8	$p + k_2(y_3 - f_{y3}) - b_8 f_{c7} F_4^2 - a_8 F_4^2 - p = 0$
e_9	$y_5 + d_{12} - 1 + f_{p2}^2 z_{p2} - a_{11} F_5^2 - a_{10} F_5^2 - f_{y5} - p = 0$
e_{10}	$p + g_{27} - b_4 F_6^2(f_{c3} + z_{u3}) - a_9 F_6^2 - p = 0$
e_{11}	$y_5 - b_6 F_7^2(f_{c5} + z_{u5}) - f_{y5} - g_{21} - p = 0$
e_{12}	$f_{y6} + y_5 - b_5 F_8^2(f_{c4} + z_{u4}) - y_6 - f_{y5} = 0$
e_{13}	$y_6 - a_{13} F_9^2 - f_{y6} - g_{23} - p = 0$
e_{14}	$y_6 - b_7 F_{10}^2(f_{c6} + z_{u6}) - a_{14} F_{10}^2 - a_{12} F_{10}^2 - f_{y6} - p = 0$
e_{15}	$f_{y4} + \frac{c_6 F_6 + F_4 x_4}{F_4 + F_6} - y_4 = 0$
e_{16}	$-1 + \frac{d_{21} F_3^2}{(-1+f_{p1})^2} + z_{p1}^2 = 0$
e_{17}	$-1 + \frac{d_{22} F_5^2}{(-1+f_{p2})^2} + z_{p2}^2 = 0$
e_{18}	$-1 + f_{u1} + u_1^2 z_{u1} = 0$
e_{19}	$-1 + f_{u2} + u_2^2 z_{u2} = 0$
e_{20}	$-1 + f_{u3} + u_3^2 z_{u3} = 0$
e_{21}	$-1 + f_{u4} + u_4^2 z_{u4} = 0$
e_{22}	$-1 + f_{u5} + u_5^2 z_{u5} = 0$
e_{23}	$-1 + f_{u6} + u_6^2 z_{u6} = 0$
e_{24}	$F_4 + F_6 - F_5 = 0$
e_{25}	$F_5 - F_8 - F_7 = 0$
e_{26}	$F_8 - F_{10} - F_9 = 0$
e_{27}	$f_{y1} + x_1 - y_1 = 0$
e_{28}	$f_{y2} + x_2 - y_2 = 0$
e_{29}	$f_{y3} + x_3 - y_3 = 0$

sensor 1, 2, and 3. Furthermore there are 21 equations, one for each fault expressed as $\dot{f}_i = 0$. Using the last equation $e_{29} : f_{y3} + x_3 - y_3 = 0$ as an example it will be shown how the behavioral mode assumptions of the equations are obtained. Since only single faults are considered in this example the process is not divided into components. Let the behavioral mode, when $f_{y3} \neq 0$, be denoted $sys = \mathbf{F}_{y3}$. Now, equation e_{29} can deduce two different equations in the following ways:

$$\begin{aligned} \text{ass}\,(x_3 - y_3 = 0) &= \mathcal{B} \backslash \{\mathbf{F}_{y3}\} \\ \text{ass}\,(f_{y3} + x_3 - y_3 = 0) &= \mathcal{B} \end{aligned}$$

Considering only single faults each basic equation including n fault variables will give rise to $n + 1$ equations. Writing the diagnostic model in the same form as

Table 9.2 *The type of each variable.*

type of variable	variable
X_u	$F_3, F_4, F_5, F_8, F_{10}, x_1, \dot{x}_1, x_2, \dot{x}_2, x_3, \dot{x}_3, x_4, \dot{x}_4, z_{u1},$ $z_{u2}, z_{u3}, z_{u4}, z_{u5}, z_{u6}, p$
F	$f_{y1}, f_{y2}, f_{y3}, f_{y4}, f_{y5}, f_{y6}, f_{u1}, f_{u2}, f_{u3}, f_{u4}, f_{u5},$ $f_{u6}, f_{c1}, f_{c2}, f_{c3}, f_{c4}, f_{c5}, f_{c6}, f_{c7}, f_{p1}, f_{p2}$
Z	$F_1, F_2, F_6, F_7, F_9, y_1, y_2, y_3, y_4, y_5, y_6, z_{p1}, z_{p2}, u_1,$ u_2, u_3, u_4, u_5, u_6

for example the water tank example in Table 2.2 the number of equations are $29 + 34 + 21 = 84$. However, since it is easy to generate the diagnostic model using this more compact form, the compact form will be used without loss of information.

The structural model for the model on the compact form in Table 9.1 can be viewed in Figure 9.2. The circles denote that the corresponding variable is non-linearly included. Three different types of variables are defined in the compact form: the unknown variables X_u, the variables F describing faults, and the known variables Z. The variables in the model shown in Table 9.1 are divided into three types of variables as defined in Table 9.2.

9.3 Extracting the No-Fault Model

The extracted structural model $M_{\mathbf{NF}}^{DSSM}$ has the structure shown in Figure 9.2 where the columns representing faults are deleted.

9.4 Differentiating the Model

The highest order of derivatives $m(z)$ that is known for all known variables are assumed to be one. Algorithm 8.4 is applied to the structural model in Figure 9.2. The result is that all equations except equation e_1, e_2, e_3, and e_4 are differentiated. This results in additionally 25 differentiated equations shown in Figure 9.3. Note how the knowledge concerning linear dependence influences the structural model in Figure 9.3 by comparing it with the original structural model in 9.2. For example, x_3 is linearly contained in e_{29}, hence $\mathrm{var}_{X_u} \dot{e}_{29} = \{\dot{x}_3\}$ and z_{u1} is non-linearly contained in e_6 and then follows that $\mathrm{var}_{X_u} \dot{e}_6 = \{z_{u1}, \dot{z}_{u1}\}$.

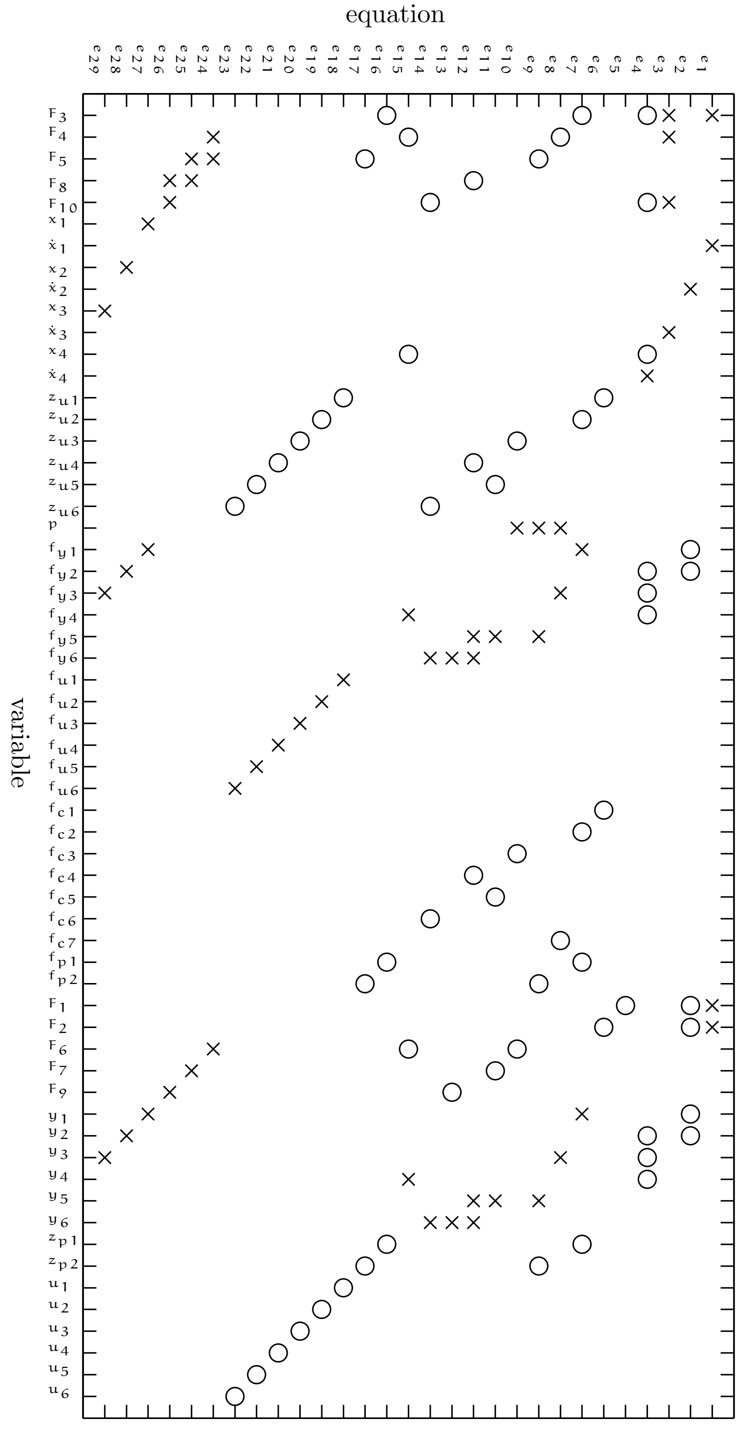

Figure 9.2 *Structural model for the set of equations in Table 9.1. The circles denote that the corresponding variable is non-linearly included and the crosses denote linearly dependent variables.*

9.5 Simplifying the Model

In the first step of simplification applied to the model in Figure 9.3, the equations $\{e_{27}, e_{28}, e_{29}\}$ are not included in the structurally overdetermined part.

The second part of the simplification finds that the variables $\dot{F}_3$, $\dot{F}_{10}$, x_1, x_2, x_3, x_4, $\dot{x}_4$, $\dot{z}_{u1}$, $\dot{z}_{u2}$, $\dot{z}_{u3}$, $\dot{z}_{u3}$, $\dot{z}_{u4}$, and $\dot{z}_{u5}$ can be eliminated. The equations that form models are $\{\dot{e}_7, \dot{e}_{16}, \dot{e}_{19}\}$, $\{\dot{e}_{14}, \dot{e}_{23}, \dot{e}_{26}\}$, $\{e_1, \dot{e}_{27}\}$, $\{e_2, \dot{e}_{28}\}$, $\{e_3, \dot{e}_{29}\}$, $\{e_4, e_{15}, \dot{e}_{15}\}$, $\{\dot{e}_6, \dot{e}_{18}\}$, $\{\dot{e}_{10}, \dot{e}_{20}\}$, $\{\dot{e}_{12}, \dot{e}_{21}\}$, and $\{\dot{e}_{11}, \dot{e}_{22}\}$. The simplified structural model is showed in Figure 9.4. Note the simplification of the model by comparing Figure 9.3 and Figure 9.4. The simplification reduces the model from 54 equations to 38 equations and reduces the unknown variables from 32 to 16.

To give an example of the reduction of the computational complexity using this merging step, Algorithm 8.6 is applied to the structural model in Figure 9.3 with and without first using the merging step. The number of times the algorithm asks for a row or a column in the structural model is computed. The result is that the merging step requires 88 calls and Algorithm 8.6 requires 335,107 calls. When Algorithm 8.6 was directly applied to the structural model in Figure 9.3 it used 1,872,753 calls. This result indicates that merging step is cheap and decreases the computational complexity of Algorithm 8.6 considerably. The next step is to find all MSS sets in the simplified model.

9.6 Finding MSS Sets

Algorithm 8.6 is applied to the simplified model. The algorithm returns 35770 MSS sets which are contained in the simplified model. The five smallest MSS sets are $\{e_5\}$, $\{\dot{e}_5\}$, $\{e_{13}\}$, $\{\dot{e}_{13}\}$, and $\{e_2, \dot{e}_{28}\}$. The largest MSS sets consist of 23 equations. In step (g) in Algorithm 8.2 the isolability of the MSS sets found is analyzed.

9.7 Evaluating Isolability

The structural isolability of the MSS sets found can be seen in the isolability matrix in Figure 9.5. All faults are detectable with the MSS sets found in the previous step. The fault mode $\mathbf{F}_{ui}$ is not isolable from $\mathbf{F}_{ci}$ where $i \in \{1, 2, 3, 4, 5, 6\}$, i.e. a constant offset in the actuator signal to valve i can always be explained as clogging in valve i. Moreover $\mathbf{F}_{y4}$ is not isolable from $\mathbf{F}_{y2}$ and $\mathbf{F}_{y3}$. Finally $\mathbf{F}_{u2}$ and $\mathbf{F}_{c2}$ are not isolable from $\mathbf{F}_{p1}$.

9.8 Extracting and Analyzing Fault Models

Since we are interested in computing the maximum isolability considering only single faults the desired isolability is chosen to be

$$\mathcal{I}_d = \{(b_i, b_j) | b_i \in \mathcal{B}_1, b_j \neq b_i, b_j \in \mathcal{B}_1 \cup \{\mathbf{NF}\}\}$$

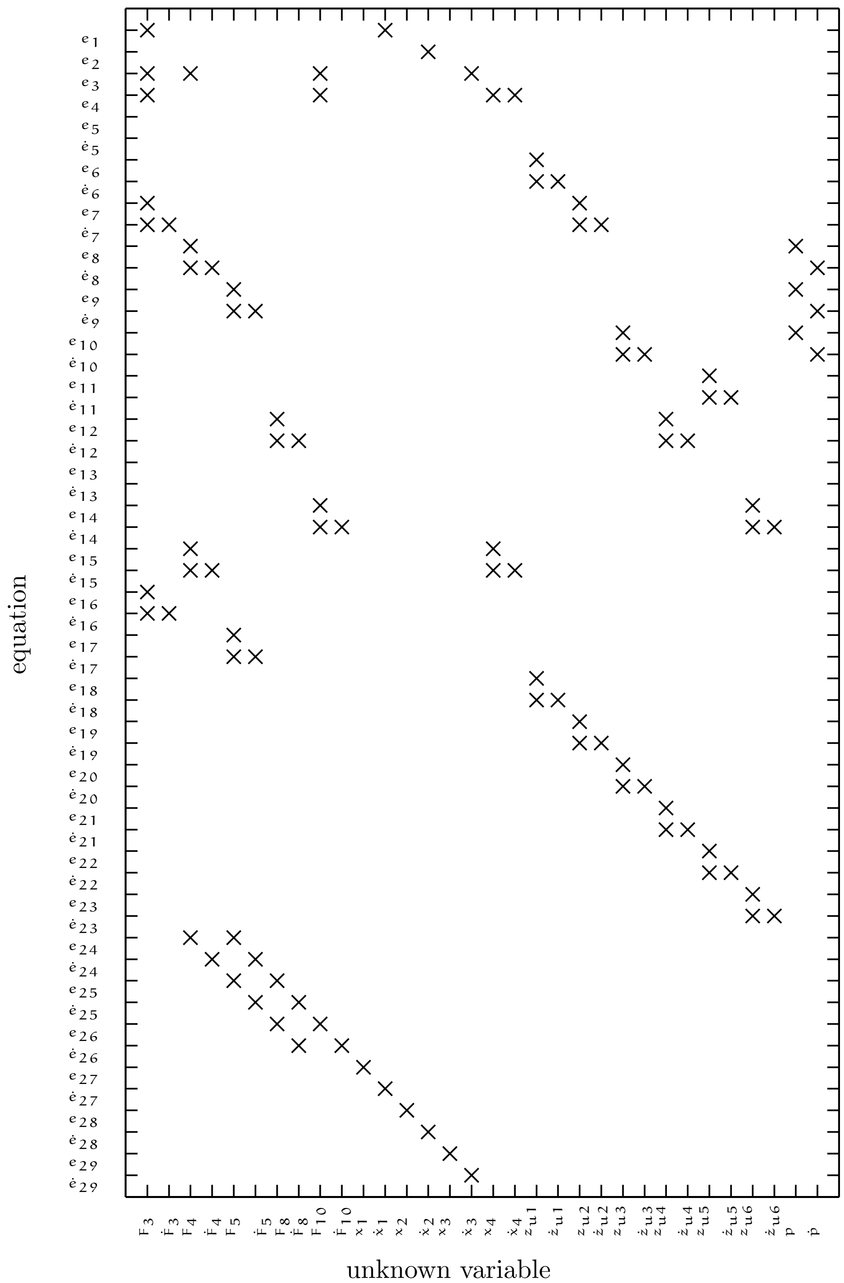

Figure 9.3 *The resulting structural model when the differentiation step is applied to the structural model in Figure 9.2. The variables* F *and* Y *are not shown. Differentiated equations are denoted with a dot after the number.*

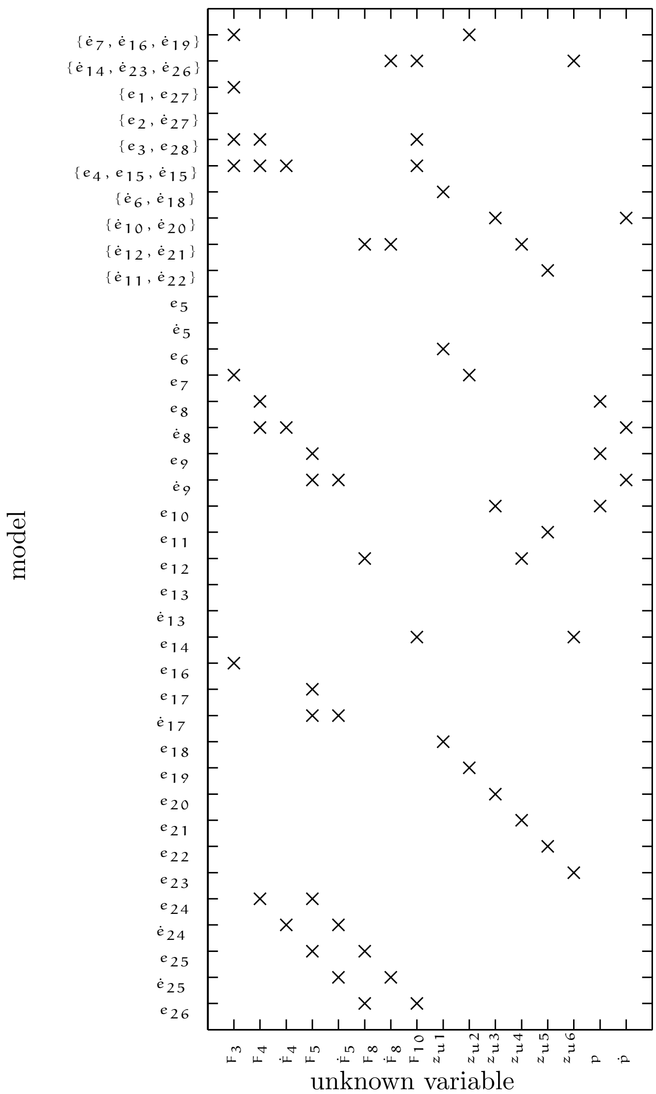

Figure 9.4 *The simplified structural model.*

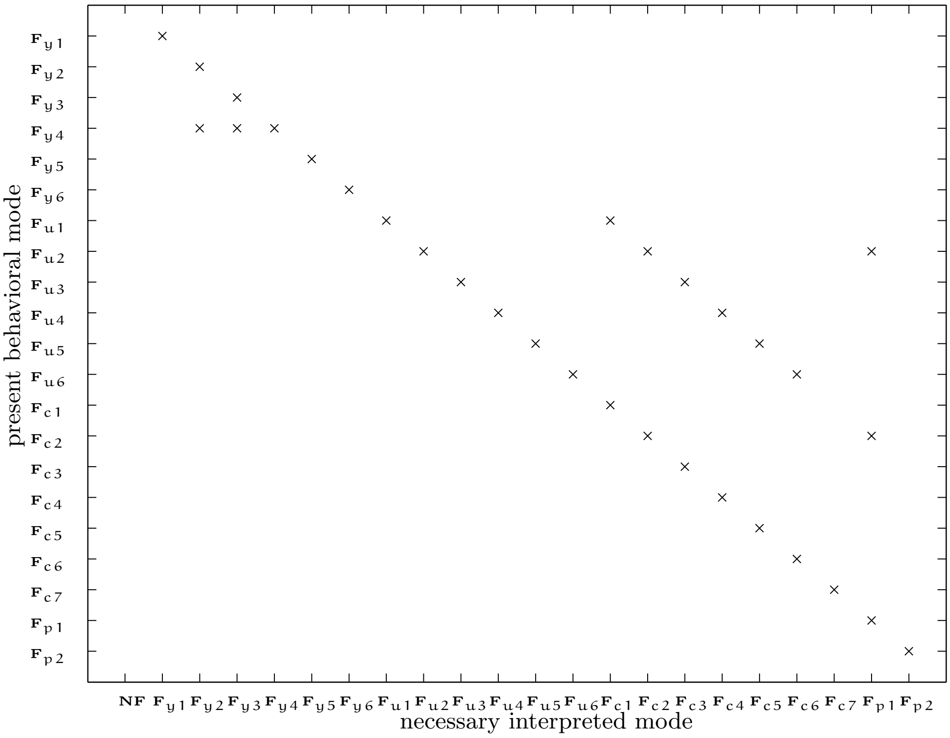

Figure 9.5 *The isolability matrix of the MSS sets corresponding to Figure 9.4*

where $\mathcal{B}_1$ denotes the single-fault modes. In the isolability matrix $I_s^{\mathbb{M}}(\text{mss}M_{\mathbf{NF}}^{DSSM})$, shown in Figure 9.5, the fault models corresponding to columns that have non-diagonal entries, i.e.

$$\{b_j|(b_i, b_j) \in \mathcal{I}_d \backslash \mathcal{I}_s^{\Delta}\}$$

are one by one analyzed. This set contains the fault modes $\mathbf{F}_{y2}$, $\mathbf{F}_{y3}$, $\mathbf{F}_{c1}$, $\mathbf{F}_{c2}$, $\mathbf{F}_{c3}$, $\mathbf{F}_{c4}$, $\mathbf{F}_{c5}$, $\mathbf{F}_{c6}$, and $\mathbf{F}_{p1}$. To see if a fault model contains MSS sets the fault is *decoupled*, i.e. considered as an unknown variable. Algorithm 8.1 step (a) is first applied to the behavioral model $M_{\mathbf{F}_{y2}}$. The goal is to find an MSS set that decouples, i.e. is insensitive for, f_{y2} and is sensitive to fault f_{y4}. An MSS set with this property increases the isolability because it gives the possibility to isolate fault mode $\mathbf{F}_{y4}$ from $\mathbf{F}_{y2}$. This implies that the cross in the row corresponding to $\mathbf{F}_{y4}$ and the column corresponding to $\mathbf{F}_{y2}$ in the structural isolability matrix in Figure 9.5 is removed. The result of decoupling fault f_{y2} is that 26,959 new MSS sets are found. The smallest MSS set of these new MSS sets, isolating fault mode $\mathbf{F}_{y4}$ from $\mathbf{F}_{y2}$ is $\{e_2, e_3, e_4, e_{15}, \dot{e}_{15}, e_{16}, e_{17}, \dot{e}_{17}, e_{24}, \dot{e}_{24}, \dot{e}_{28}, \dot{e}_{29}\}$.

Table 9.3 *The smallest MSS set with the structural isolability defined by the two first columns.*

decoupled fault	sensitive to fault	smallest MSS set with desired property
f_{y2}	f_{y4}	$\{e_2, e_3, e_4, e_{15}, \dot{e}_{15}, e_{16}, e_{17}, \dot{e}_{17}, e_{24}, \dot{e}_{24}, \dot{e}_{28}, \dot{e}_{29}\}$
f_{y3}	f_{y4}	$\{e_4, e_8, e_9, e_{14}, e_{15}, \dot{e}_{15}, e_{16}, e_{17}, \dot{e}_{17}, e_{23}, e_{24}, \dot{e}_{24}\}$
f_{c1}	f_{u1}	$\{e_6, \dot{e}_6, e_{18}, \dot{e}_{18}\}$
f_{c2}	f_{u2}	$\{e_7, \dot{e}_7, e_{16}, \dot{e}_{16}, e_{19}, \dot{e}_{19}\}$
f_{c3}	f_{u3}	$\{e_9, \dot{e}_9, e_{10}, \dot{e}_{10}, e_{17}, \dot{e}_{17}, e_{20}, \dot{e}_{20}\}$
f_{c4}	f_{u4}	$\{e_{12}, \dot{e}_{12}, e_{17}, \dot{e}_{17}, e_{21}, \dot{e}_{21}, e_{25}, \dot{e}_{25}\}$
f_{c5}	f_{u5}	$\{e_{11}, \dot{e}_{11}, e_{22}, \dot{e}_{22}\}$
f_{c6}	f_{u6}	$\{e_{12}, \dot{e}_{12}, e_{14}, \dot{e}_{14}, e_{21}, \dot{e}_{21}, e_{23}, \dot{e}_{23}, e_{26}, \dot{e}_{26}\}$
f_{p1}	f_{u2}	$\{e_1, e_7, e_{16}, e_{19}, \dot{e}_{27}\}$

Table 9.4 *The first 6 MSS sets in the reordered list.*

	MSS	
1	e_5	
2	$\dot{e}_5$	
3	e_{13}	
4	$\dot{e}_{13}$	
5	e_2	$\dot{e}_{28}$
6	e_6	e_{18}

Next also the fault modes $\mathbf{F}_{y3}$, $\mathbf{F}_{c1}$, $\mathbf{F}_{c2}$, $\mathbf{F}_{c3}$, $\mathbf{F}_{c4}$, $\mathbf{F}_{c5}$, $\mathbf{F}_{c6}$, and $\mathbf{F}_{p1}$ are used as inputs to Algorithm 8.1 step (a). To show some results the smallest MSS set for each desired isolability property is shown in Table 9.3. With those additional MSS sets all faults are detectable and isolable. The next step is to select a small subset of all found MSS sets that have maximum isolability.

9.9 Selecting a Small Subset of MSS Sets

The number n_m defined in Section 8.7 is here chosen as the number of the 29 basic equations included in MSS number m. Note that the additional equations $\dot{f}_i = 0$ are not counted in n_m. First the MSS sets are reordered in increasing size of n_m. The first 6 MSS sets in the reordered list are shown in Table 9.4

Then the algorithm selects those MSS sets that increase the isolability starting from the MSS sets with smallest n_m in Table 9.4. According to Figure 9.2 neither

Table 9.5 *The 36 selected MSS sets with full structural isolability considering only single-faults.*

	MSS sets											
1	e_{13}											
2	e_{2}	$\dot{e}_{28}$										
3	e_{6}	e_{18}										
4	e_{11}	e_{22}										
5	e_{1}	e_{16}	$\dot{e}_{27}$									
6	e_{6}	$\dot{e}_{6}$	$\dot{e}_{18}$									
7	e_{11}	$\dot{e}_{11}$	$\dot{e}_{22}$									
8	e_{11}	e_{22}	$\dot{e}_{22}$									
9	e_{7}	e_{16}	e_{19}									
10	e_{8}	e_{9}	e_{17}	e_{24}								
11	e_{9}	e_{10}	e_{17}	e_{20}								
12	e_{12}	e_{17}	e_{21}	e_{25}								
13	e_{6}	$\dot{e}_{6}$	e_{18}	$\dot{e}_{18}$								
14	e_{11}	$\dot{e}_{11}$	e_{22}	$\dot{e}_{22}$								
15	e_{7}	$\dot{e}_{7}$	e_{16}	$\dot{e}_{16}$	$\dot{e}_{19}$							
16	$\dot{e}_{7}$	e_{16}	$\dot{e}_{16}$	e_{19}	$\dot{e}_{19}$							
17	e_{8}	e_{10}	e_{17}	e_{20}	e_{24}							
18	e_{12}	e_{14}	e_{21}	e_{23}	e_{26}							
19	e_{14}	e_{17}	e_{23}	e_{25}	e_{26}							
20	e_{1}	e_{7}	e_{16}	e_{19}	$\dot{e}_{27}$							
21	$\dot{e}_{8}$	$\dot{e}_{9}$	e_{17}	$\dot{e}_{17}$	e_{24}	$\dot{e}_{24}$						
22	e_{7}	$\dot{e}_{7}$	e_{16}	$\dot{e}_{16}$	e_{19}	$\dot{e}_{19}$						
23	e_{9}	$\dot{e}_{9}$	e_{10}	$\dot{e}_{10}$	e_{17}	$\dot{e}_{17}$	$\dot{e}_{20}$					
24	e_{12}	$\dot{e}_{12}$	e_{17}	$\dot{e}_{17}$	$\dot{e}_{21}$	e_{25}	$\dot{e}_{25}$					
25	e_{8}	e_{10}	e_{12}	e_{20}	e_{21}	e_{24}	e_{25}					
26	$\dot{e}_{12}$	e_{17}	$\dot{e}_{17}$	e_{21}	$\dot{e}_{21}$	e_{25}	$\dot{e}_{25}$					
27	e_{9}	$\dot{e}_{9}$	e_{10}	$\dot{e}_{10}$	e_{17}	$\dot{e}_{17}$	e_{20}	$\dot{e}_{20}$				
28	e_{12}	$\dot{e}_{12}$	e_{17}	$\dot{e}_{17}$	e_{21}	$\dot{e}_{21}$	e_{25}	$\dot{e}_{25}$				
29	e_{12}	$\dot{e}_{12}$	e_{14}	$\dot{e}_{14}$	e_{21}	$\dot{e}_{21}$	$\dot{e}_{23}$	e_{26}	$\dot{e}_{26}$			
30	e_{14}	e_{17}	$\dot{e}_{17}$	e_{23}	$\dot{e}_{23}$	e_{25}	$\dot{e}_{25}$	e_{26}	$\dot{e}_{26}$			
31	e_{3}	e_{4}	e_{15}	$\dot{e}_{15}$	e_{16}	e_{17}	e_{24}	$\dot{e}_{24}$	$\dot{e}_{29}$			
32	e_{12}	$\dot{e}_{12}$	e_{14}	$\dot{e}_{14}$	e_{21}	$\dot{e}_{21}$	e_{23}	$\dot{e}_{23}$	e_{26}	$\dot{e}_{26}$		
33	e_{1}	e_{3}	e_{4}	e_{15}	$\dot{e}_{15}$	e_{17}	$\dot{e}_{17}$	e_{24}	$\dot{e}_{24}$	$\dot{e}_{27}$	$\dot{e}_{29}$	
34	e_{3}	e_{4}	e_{8}	$\dot{e}_{8}$	e_{10}	$\dot{e}_{10}$	e_{15}	$\dot{e}_{15}$	e_{16}	e_{20}	$\dot{e}_{20}$	$\dot{e}_{29}$
35	e_{4}	e_{8}	e_{9}	e_{14}	e_{15}	$\dot{e}_{15}$	e_{16}	e_{17}	$\dot{e}_{17}$	e_{23}	e_{24}	$\dot{e}_{24}$
36	e_{2}	e_{3}	e_{4}	e_{15}	$\dot{e}_{15}$	e_{16}	e_{17}	$\dot{e}_{17}$	e_{24}	$\dot{e}_{24}$	$\dot{e}_{28}$	$\dot{e}_{29}$

the first $\{e_5\}$ nor the second MSS set $\{\dot{e}_5\}$ is sensitive to any fault and is therefore not selected. The third MSS set $\{e_{13}\}$ is sensitive only to f_{y6}, i.e. $\mathbf{F}_{y6}$ can be detected and isolated. The isolability is improved with this MSS set and therefore it is selected. The fourth MSS set is not sensitive for any fault and is therefore not selected. The 5th MSS set can be sensitive for f_{y1} and f_{y2} and therefore isolates $\mathbf{F}_{y1}$ and $\mathbf{F}_{y2}$ from all other fault modes. This MSS set is selected. When all MSS sets have been analyzed, 36 MSS sets, shown in Table 9.5, are selected.

The first three MSS sets can be recognized from Table 9.4. The equations describing fault models are not explicitly written in Table 9.5. In Figure 9.6 the sequence of isolability matrices, defined by adding one MSS at a time from Table 9.5, is shown. In Figure 9.7 it is shown how the desired isolability fulfillment increases adding the selected MSS sets once at a time. In Figure 9.7 the desired isolability fulfillment is instead plotted against the number n_m of the selected MSS sets. Note that most of the desired isolability is obtained when only the MSS sets with $n_m \leq 5$ are used.

From the 36 MSS sets the *incidence* matrix in Figure 9.9 is obtained. An empty entry in position (i, j) in the incidence matrix denotes that fault j cannot invalidate the model corresponding to MSS set i. The influence matrix therefore defines Φ_i as the set of behavioral modes that correspond to columns with empty entries in row i. Note that $\mathbf{NF}$ is included in all Φ_i in this example. Hence it is impossible

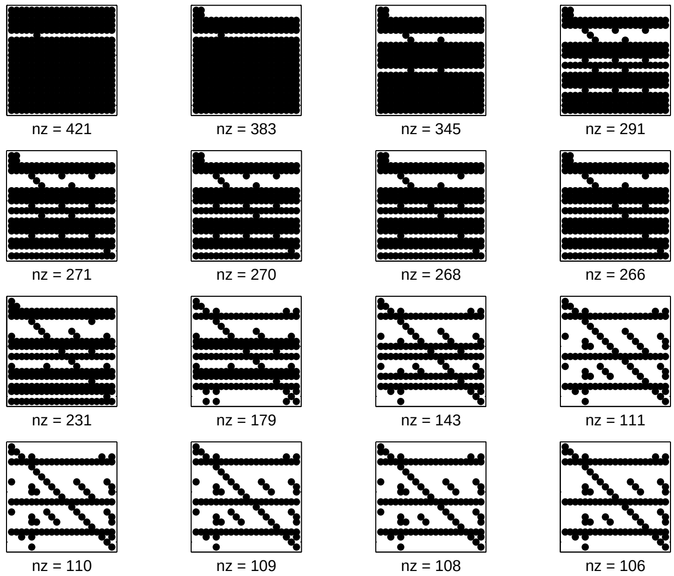

Figure 9.6 *The sequence of the first 16 isolability matrices defined by adding one MSS at a time from Table 9.5. The numbers denote the number of crosses in each fault matrix.*

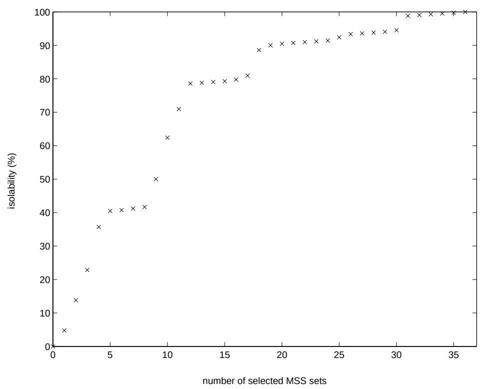

Figure 9.7 *The labels on the x-axis indicate how many of the first selected MSS sets in Table 9.5 that are used. The y-axis shows desired isolability fulfillment with those MSS sets.*

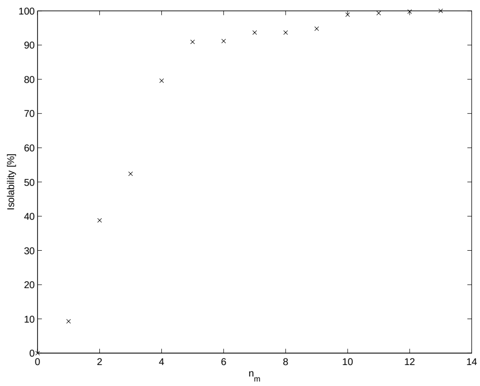

Figure 9.8 *The number* n_m *plotted against the desired isolability fulfillment.*

to conclude that the process is working normally.

9.10 Generating Consistency Relations

In this example consistency relations are used to validate the MSS sets. However, other methods have been presented that can be used to validate the MSS sets, e.g. observers. The consistency relations corresponding to the MSS sets are calculated, by using the function `Eliminate` in Mathematica. Most of the equations in the model are polynomial equations. For polynomial equation-systems, the function Eliminate uses Gröbner Basis for elimination.

All MSS sets with 7 or less equations were easily eliminated and consistency relation were obtained. The consistency relations from the MSS set 23, 24, 25 and 26 were obtained from the elimination function, but were not useful because of bad numerical properties. However, small MSS sets make the largest contribution to the isolability. To see this, Figure 9.8 shows the percentage of the desired isolability fulfillment when only the first n selected MSS sets in Table 9.5 are used. The number n is plotted on the x-axis. It is clear that the structural isolability reduces slightly, without using large MSS sets, difficult to calculate.

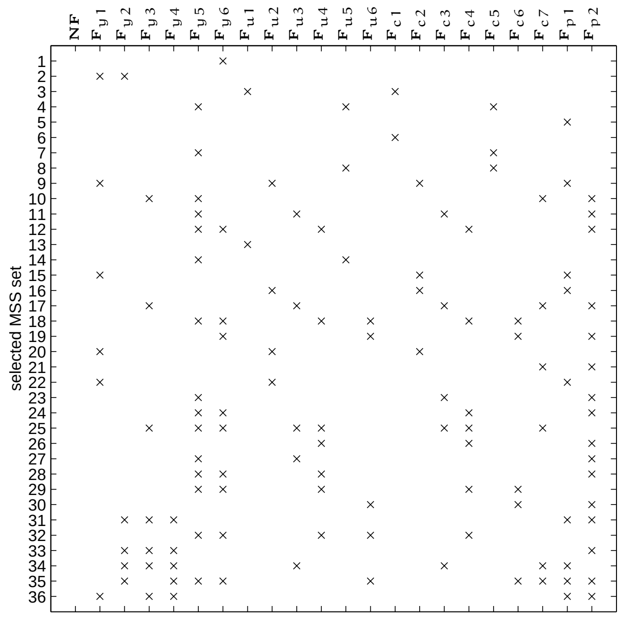

Figure 9.9 *The incidence matrix of the selected MSS sets corresponding to Figure 9.4*

Finally a few examples of consistency relations derived by Mathematica are

MSS	consistency relations with faults
1	$y_6 - a_1 - a_2 F_9{}^2 = f_{y6}$
2	$F_1(-a_3 + a_4 y_2) + F_2(-a_5 + a_4 y_2) + a_6 y_1 \dot{y}_2 =$
	$= a_6 f_{y1} \dot{y}_2 + a_4 F_1 f_{y2} + a_4 F_2 f_{y2}$
5	$f_{p1} \neq 1 \wedge$
	$a_7 + a_9 F_1 \dot{y}_1 + a_9 F_2 \dot{y}_1 - a_{10} F_1{}^2 - a_{11} F_1 F_2 - a_{10} F_2{}^2 - a_7 z_{p1}{}^2 - a_{12} \dot{y}_1^2 =$
	$= -a_7 f_{p1}{}^2 - a_8 f_{p1} z_{p1}{}^2 + a_8 f_{p1} + +a_7 f_{p1}{}^2 z_{p1}{}^2$
6	$u_1 \big(F_2(a_{13} - a_{15} F_2{}^2)\dot{u}_1 - (a_{13} + a_{14} F_2{}^2) u_1 \dot{F}_2\big) =$
	$= f_{u1}{}^2(-a_{13} - a_{14}\, F_2{}^2)\, \dot{F}_2 +$
	$f_{u1}\big((-2a_{13} - 2a_{14} F_2{}^2) u_1 \dot{F}_2 + F_2(a_{13} + (-a_{15} - a_{16} f_{c1}) F_2{}^2)\dot{u}_1\big)$

The computational form of these consistency relations are

MSS	computational form of some consistency relations
1	$y_6 - a_1 - a_2 F_9{}^2 = 0$
2	$F_1(-a_3 + a_4 y_2) + F_2(-a_5 + a_4 y_2) + a_6 y_1 \dot{y}_2 = 0$
5	$a_7 + a_9 F_1 \dot{y}_1 + a_9 F_2 \dot{y}_1 - a_{10} F_1{}^2 - a_{11} F_1 F_2 - a_{10} F_2{}^2 - a_7 z_{p1}{}^2 - a_{12} \dot{y}_1^2 = 0$
6	$u_1 \big(F_2(a_{13} - a_{15} F_2{}^2)\dot{u}_1 - (a_{13} + a_{14} F_2{}^2) u_1 \dot{F}_2\big) = 0$

For some simulation results, utilizing consistency relations in this industrial example, see (Biteus 2001). Finally, a future work is to apply all design ideas and algorithms presented in this thesis to a large industrial process, e.g. by continuing the work on the paper mill.

10

Conclusions

Today many technical processes are complex and highly integrated, and it is a demanding and time-consuming task to design a diagnostic system. Different algorithms and analysis methods that help and automate the design of diagnostic systems are therefore presented in this thesis.

In a diagnostic system a number of diagnostic tests validate different testable models, i.e. different parts of the diagnostic model, with respect to observations of the process. A diagnostic system is defined as sound and complete by the requirement that for any observation exactly the same possible behavioral modes are given from the diagnostic system as the behavioral modes that together with the observations are consistent with the diagnostic model. The presented theory and algorithms for supporting the design of diagnostic systems can be divided into three parts: computing which testable models that must be used to design sound and complete diagnostic systems, finding MSS sets, and computing isolability limitations of diagnostic models. The structural algorithm for finding MSS sets is applied to a large non-linear example, a part of a paper mill. In spite of the complexity of this process, a small set of tests with the high isolability is successfully derived.

Computing which Testable Models to Use in Diagnostic Systems

A key result of designing diagnostic systems is Theorem 5.6 where a necessary and sufficient condition for which set of models that results in a sound and complete diagnostic system if a strong test is designed for each model. Three algorithms are proposed to find a set γ of models to check for consistency, i.e. Algorithm 7.1, Algorithm 7.2, and Algorithm 7.3.

Algorithm 7.1 is the algorithm that presumes that Assumption 7.1 holds. In

Theorem 7.7 it is proven that linear static models fulfill Assumption 7.1. In Example 7.8, a specialized version of Algorithm 7.1 for linear static models is applied to a linear static model. When the models to check for consistency are computed, an additional step computes the test quantities. Hence in Example 7.8 an algorithm is developed based on Algorithm 7.1 that computes a sound and complete diagnostic system for any linear static diagnostic model.

Algorithm 7.2, computes a set γ that corresponds to sound and complete diagnostic system for any diagnostic model, i.e. without requiring Assumption 7.1. However this algorithm has disadvantages compared to Algorithm 7.1. In Algorithm 7.2 two extra steps need to be performed, i.e. the additional analytical test in step (c) and the extra structural computation in step (d) .

The last of the three algorithms, Algorithm 7.3, is the only purely structural algorithm that finds a set γ that corresponds to a sound and complete diagnostic system if Assumption 7.1 holds for the diagnostic model and if (6.109) where $\hat{M} = M_b^+$ holds for all $b \in \mathcal{B}$.

Algorithm for Finding MSS Sets

In all these three proposed algorithms for finding a γ that corresponds to a sound and complete diagnostic system, there is a common step that finds all MSS sets of equations. Algorithm 8.1 finds all MSS sets in a model described by differential-algebraic equations. Step (b) in Algorithm 8.1 handles derivatives in two different ways, of which one is a new way of handling derivatives.

Algorithms for Computing Isolability Limitations of Diagnostic Models

Another property that can be used to compare diagnostic models and diagnostic systems is their capability of distinguish behavioral modes pairwise, i.e. their isolability. Diagnostic systems applied in industry are often subject to isolability criteria.

There are two main methods to compute an upper limit of the isolability using mainly structural properties, i.e. to compute a structural isolability. The first method is to use (6.122) where $C_b = M_b$. This method is purely structural and requires no assumptions. The second method is to calculate (6.122) where C_b is chosen as described in Section 6.4.3. This method can be done if Assumption 6.1 holds.

Using these two algorithms, shortages of diagnostic models can be detected and the isolability improvements obtained from for example introducing fault models and extra sensors can easily be analyzed.

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