qpOASES - Online Active Set Strategy for Fast Linear MPC

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Linear MPC (or QP subproblems in NMPC)

For

- linear dynamic system $\qquad x_{k+1} = Ax_k + Bu_k\,,$
- linear constraints
- quadratic cost

only *quadratic program (QP)* needs to be solved:

$$
\begin{aligned}
\min_{\substack{u_0,\ldots,u_{N-1}\\ x_1,\ldots,x_N}} \quad & x_N^T P x_N + \sum_{i=0}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i\right) \\
\text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k\,, \\
& (x_0 \text{ given})\,, \\
& \underline{c} \le Cx_k \le \overline{c}\,, \\
& \underline{d} \le Du_k \le \overline{d}\,, \\
& c_T \le C_T x_N\,,
\end{aligned}
$$

Linear MPC = parametric QP

Eliminate states via "condensing", obtain smaller scale *quadratic program (QP)* in variables $w := (u_0^T, \ldots, u_{N-1})^T$

$$\mathrm{QP}(x_0): \quad \min_{w} \; \frac{1}{2} w^T H w + w^T \underbrace{F^T x_0}_{=: g(x_0)}$$
$$\text{s.t.} \quad Gw \geq \underbrace{\bar{b} + E x_0}_{=: b(x_0)},$$

(assumption: H positive definite)

QP depends on x_0 via *affine functions* $g(x_0)$ and $b(x_0)$

Karush-Kuhn-Tucker (KKT) Conditions

Theorem

Let $\mathrm{QP}(x_0)$ be a strictly convex and feasible quadratic program. Then there exists a unique $w^* \in \mathbb{R}^n$ and at least one working set $\mathbb{A}$ and a vector $y^* \in \mathbb{R}^m$ which satisfy the following conditions:

$$
\begin{aligned}
Hw^* - G_{\mathbb{A}}^T y_{\mathbb{A}}^* &= -g(x_0), \\
G_{\mathbb{A}} w^* &= b_{\mathbb{A}}(x_0), \\
y_{\mathbb{I}}^* &= 0, \;\; (\mathbb{I} := \{1, \ldots, m\} \setminus \mathbb{A}), \\
G_{\mathbb{I}} w^* &\geq b_{\mathbb{I}}(x_0), \\
y_{\mathbb{A}}^* &\geq 0.
\end{aligned}
$$

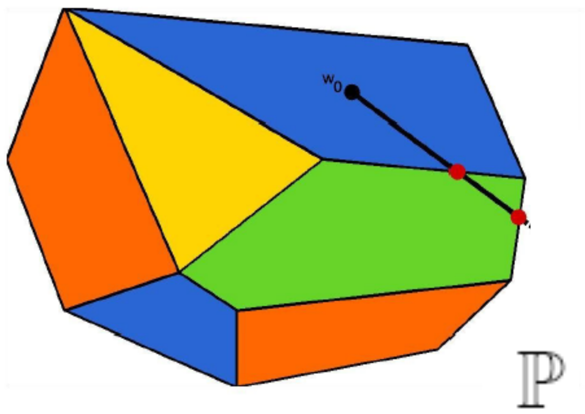
Parametric QP Solution Structure

Define set of “feasible parameters“:

$$\mathbb{P} := \{x_0 \in \mathbb{R}^{n_x} \mid \mathrm{QP}(x_0) \text{ is } \textit{feasible} \}$$

Well known:

THEOREM: Set $\mathbb{P}$ is convex, and can be partitioned into polyhedral „critical regions“ each corresponding to a different working set $\mathbb{A}$.
QP solution on each region is affine in x_0



Sketch of Proof for Polyhedral Critical Regions

Check KKT conditions for fixed working set $\mathbb{A}$:

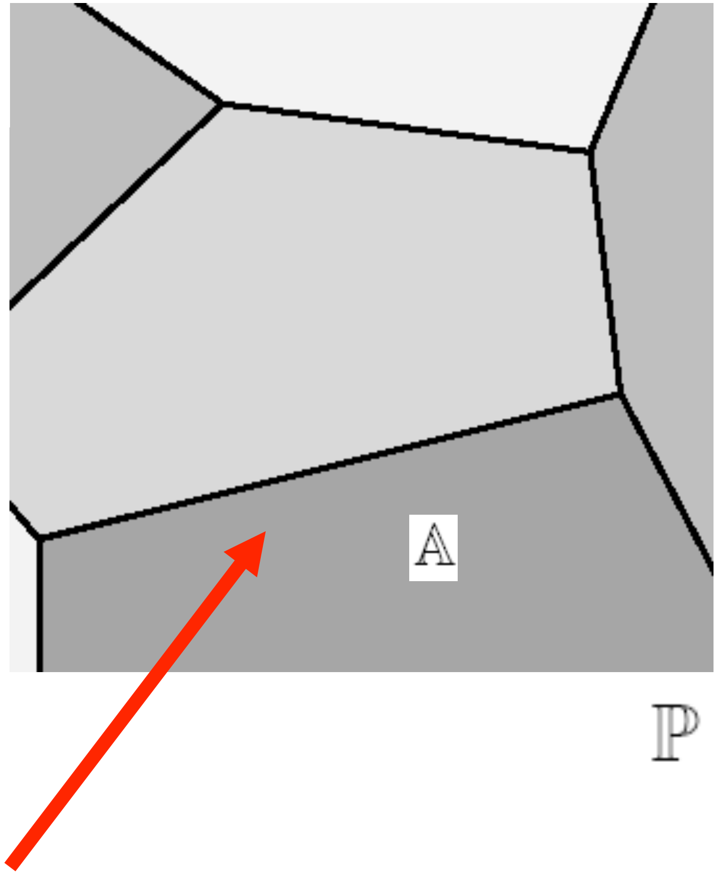
1. $g(x_0)$, $b(x_0)$ affine: then w^* , y^* affine, because solution of linear system:

$$
\begin{aligned}
Hw^* - G_{\mathbb{A}}^T y_{\mathbb{A}}^* &= -g(x_0), \\
G_{\mathbb{A}} w^* &= b_{\mathbb{A}}(x_0), \\
y_{\mathbb{I}}^* &= 0,
\end{aligned}
$$

2. w^*, y^* affine, therefore

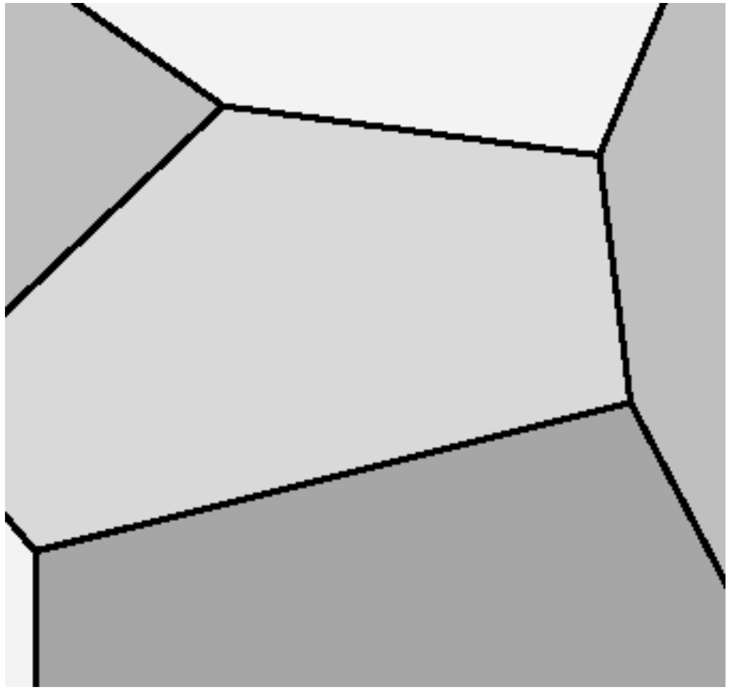
$$
\begin{aligned}
G_{\mathbb{I}} w^* &\geq b_{\mathbb{I}}(x_0), \\
y_{\mathbb{A}}^* &\geq 0.
\end{aligned}
$$

are **linear constraints** on x_0 that define polyhedral „critical region“ in $\mathbb{P}$



Explicit MPC: Precalculate Everything

Idea: Compute control on all critical regions in advance (Bemporad, Borrelli, Morari, 2002).



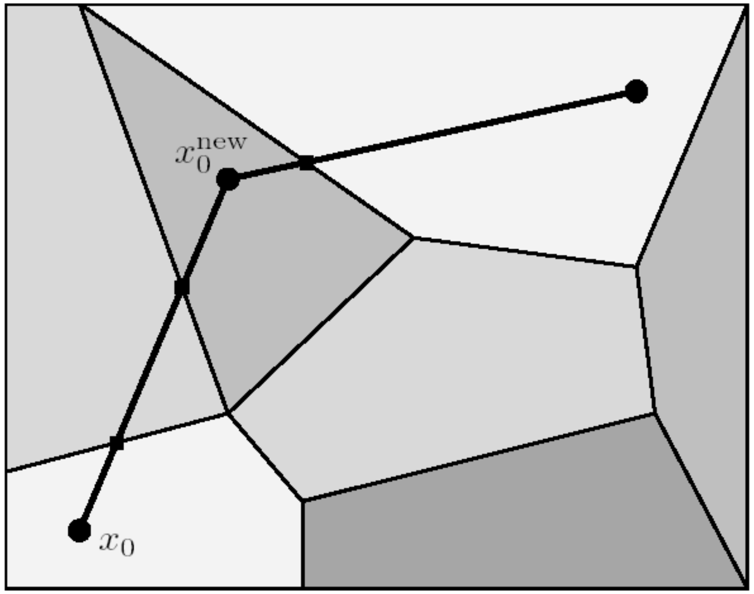
Pro: MPC in microsecs. possible
Contra: problem size limited

Example: 50 variables, lower and upper bounds:
$3^{50} = 10^{23}$ possible critical regions.
Prohibitive.

Online Active Set Strategy (qpOASES)

Combine Explicit and Online MPC

- compute affine solution only on current critical region
- go on straight line from old to new problem data ($\mathbb{P}$ convex !)
- **solve each QP on path exactly (keep primal-dual feasibility)!**
- need to change working set only at boundaries of critical regions

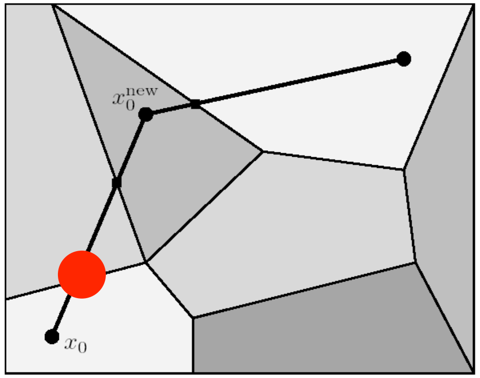


$\mathbb{P}$

How to compute each step?

- determine change in $g(x_0)$ and $b(x_0)$, solve KKT system

$$\begin{pmatrix} H & G_{\mathbb{A}}^T \\ G_{\mathbb{A}} & 0 \end{pmatrix} \begin{pmatrix} \Delta w^* \\ -\Delta y_{\mathbb{A}}^* \end{pmatrix} = \begin{pmatrix} -\Delta g \\ \Delta b_{\mathbb{A}} \end{pmatrix}$$



- choose steplength $\tau_{\max}$ maximal such that

$$G_i^T (w^* + \tau \Delta w^*) \geq b_i(x_0) + \tau \Delta b_i$$

still holds for all inactive (primal) constraints, and

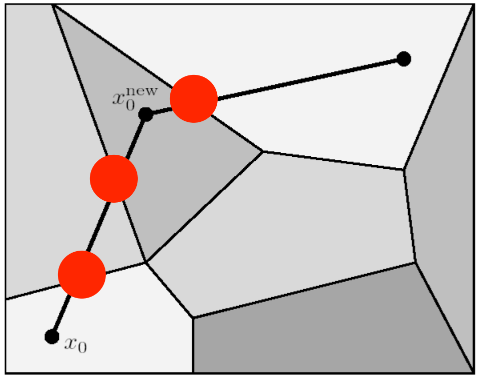
$$y_i^* + \tau \Delta y_i \geq 0$$

for all active dual variables: Set $\tau_{\max} := \min \{1, \tau_{\max}^{\text{prim}}, \tau_{\max}^{\text{dual}}\}$

with $\tau_{\max}^{\text{prim}} := \min\limits_{\substack{i \in \mathbb{I} \\ G_i^T \Delta w^* < \Delta b_i}} \dfrac{b_i(x_0) - G_i^T w^*}{G_i^T \Delta w^* - \Delta b_i}$ and $\tau_{\max}^{\text{dual}} := \min\limits_{\substack{i \in \mathbb{A} \\ \Delta y_i < 0}} -\dfrac{y_i^*}{\Delta y_i}$

How to change working set?

- add or remove constraints to/from working set when crossing borders of critical regions



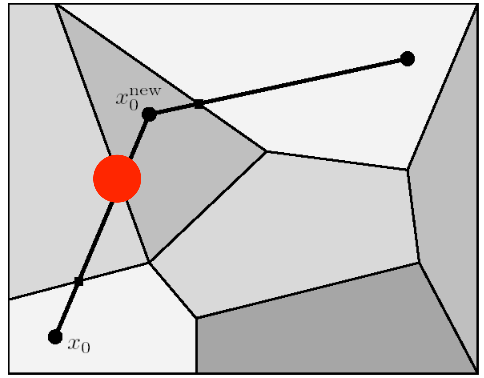
- use null space approach, keep QR-factorization of active constraint matrix, and Cholesky factorization of projected hessian:

$$G_{\mathbb{A}} = \begin{pmatrix} 0 & T \end{pmatrix} \begin{pmatrix} Z^T \\ Y^T \end{pmatrix},$$

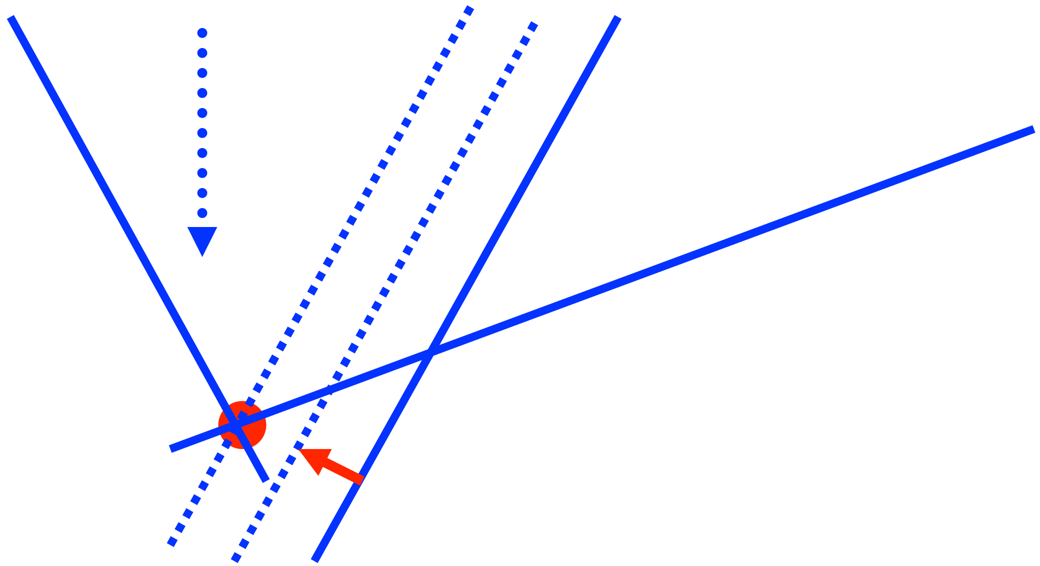
$$Z^T H Z = R^T R,$$

- each working set change costs only O(n^2) flops, exactly as one QP iteration in efficient QP solvers!

Extra difficulty: linear independence often violated



- during homotopy, often redundant constraints become active and cause degeneracy
- example of three active constraints in 2-D:



How to deal with degeneracy?

Without linear independence, KKT system becomes unsolvable! Addition of degenerate constraints must be avoided.

Remedy [Best 1996]: before adding extra row G_j to $G_{\mathbb{A}}$ solve auxiliary system

$$\begin{pmatrix} H & G_{\mathbb{A}}^T \\ G_{\mathbb{A}} & 0 \end{pmatrix} \begin{pmatrix} s \\ \xi \end{pmatrix} = \begin{pmatrix} G_j \\ 0 \end{pmatrix}$$

if $s = 0$, linear dependence is detected, and ξ helps to find a constraint from $\mathbb{A}$ that can be removed

Summary of qpOASES Algorithm

(1) Calculate Δx_0, Δg and Δb

(2) Calculate primal and dual step directions Δw^* and Δy^*

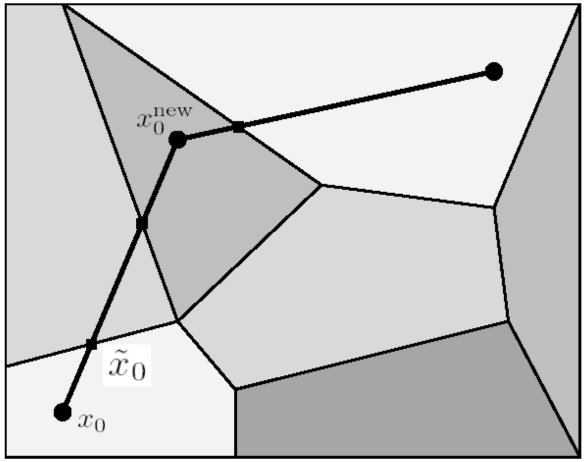
(3) Determine maximum homotopy step length $\tau_{\max} := \min\left\{1, \tau_{\max}^{\text{prim}}, \tau_{\max}^{\text{dual}}\right\}$

(4) Obtain optimal solution of $\text{QP}(\tilde{x}_0)$:

(a) $\tilde{x}_0 \leftarrow x_0 + \tau_{\max}\Delta x_0$,

(b) $\tilde{w}^* \leftarrow w^* + \tau_{\max}\Delta w^*$,

(c) $\tilde{y}^* \leftarrow y^* + \tau_{\max}\Delta y^*$.



(5) `if` $\tau_{\max} = 1$:

Optimal solution of $\text{QP}(x_0^{\text{new}})$ found.

(6) `if` $\tau_{\max} = \tau_{\max}^{\text{dual}}$:

Remove a dual blocking constraint j $\left(\tau_{\max}^{\text{dual}} = -\frac{y_j^*}{\Delta y_j}\right)$ from working set,

`elseif` $\tau_{\max} = \tau_{\max}^{\text{prim}}$:

Add a primal blocking constraint j $\left(\tau_{\max}^{\text{prim}} = \frac{b_j(x_0) - G_j^T w^*}{G_j^T \Delta w^* - \Delta b_j}\right)$

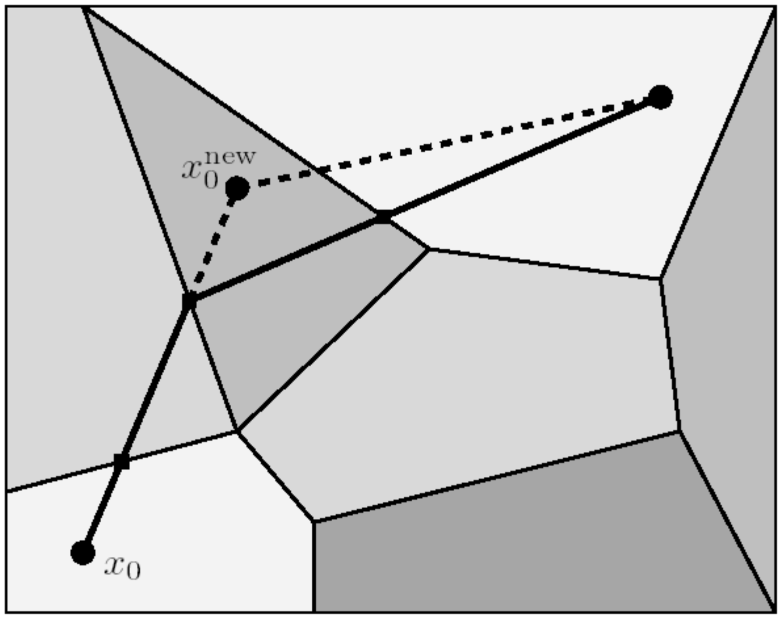
ensuring linear independence

(7) Set $x_0 \leftarrow \tilde{x}_0$, $w^* \leftarrow \tilde{w}^*$, $y^* \leftarrow \tilde{y}^*$ and continue with step (1).

Real-Time Variant of qpOASES Algorithm

Limit number of active set changes per sampling time:

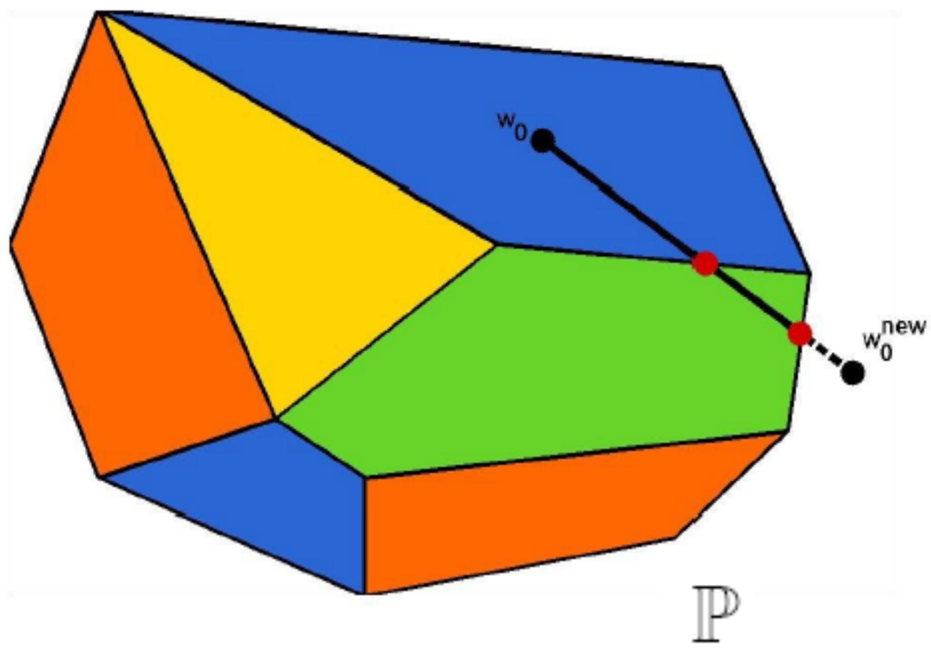
- lag behind, if too many changes necessary
- deliver solution of some problem between old and new
- make good for lag in later problems



Infeasibility treatment

$\mathbb{P}$ convex: QP on path infeasible <=> new QP infeasible

Stop at last feasible QP, wait for better posed problems



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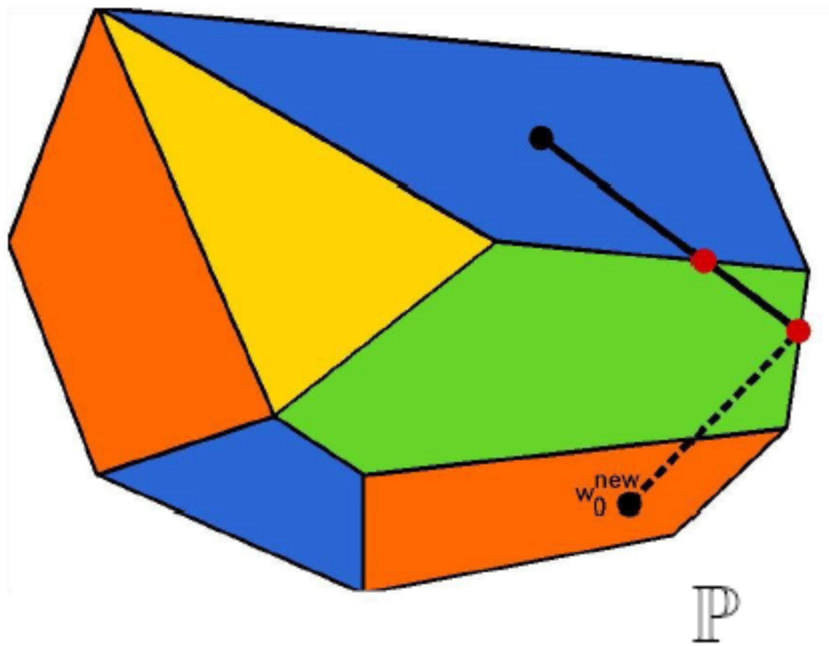
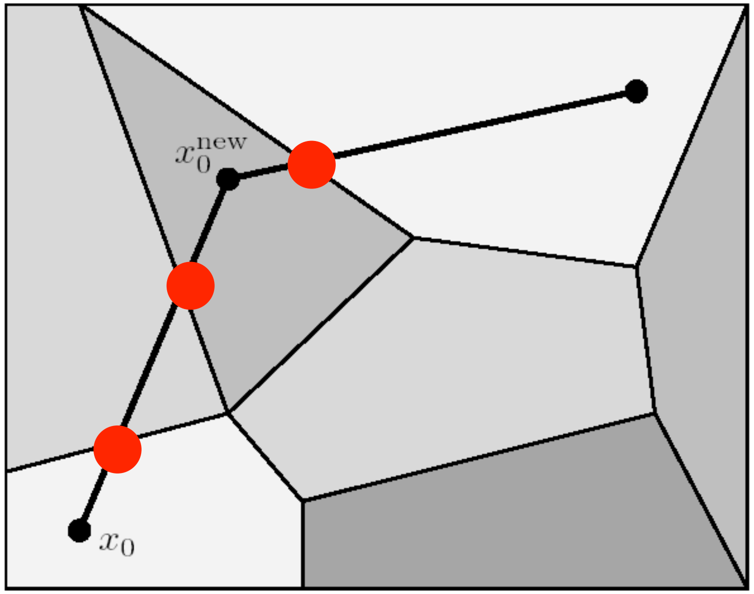


Fortunately:
new QP feasible <=> full path is feasible, and strategy works again

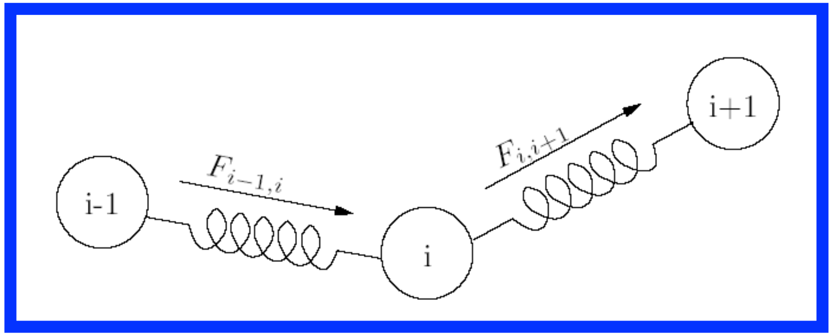
qpOASES: Open Code by Hans Joachim Ferreau



qpOASES:
open source C++ code by Hans Joachim Ferreau

http://www.kuleuven.be/optec/software/qpOASES

Application to Chain of Masses



- 10 balls connected by springs, No. 1 fixed
- 3-D velocity of ball No. 10 controlled: $\dot{x}_{N+1} = u(t)$
- 2nd order ODE for other balls:

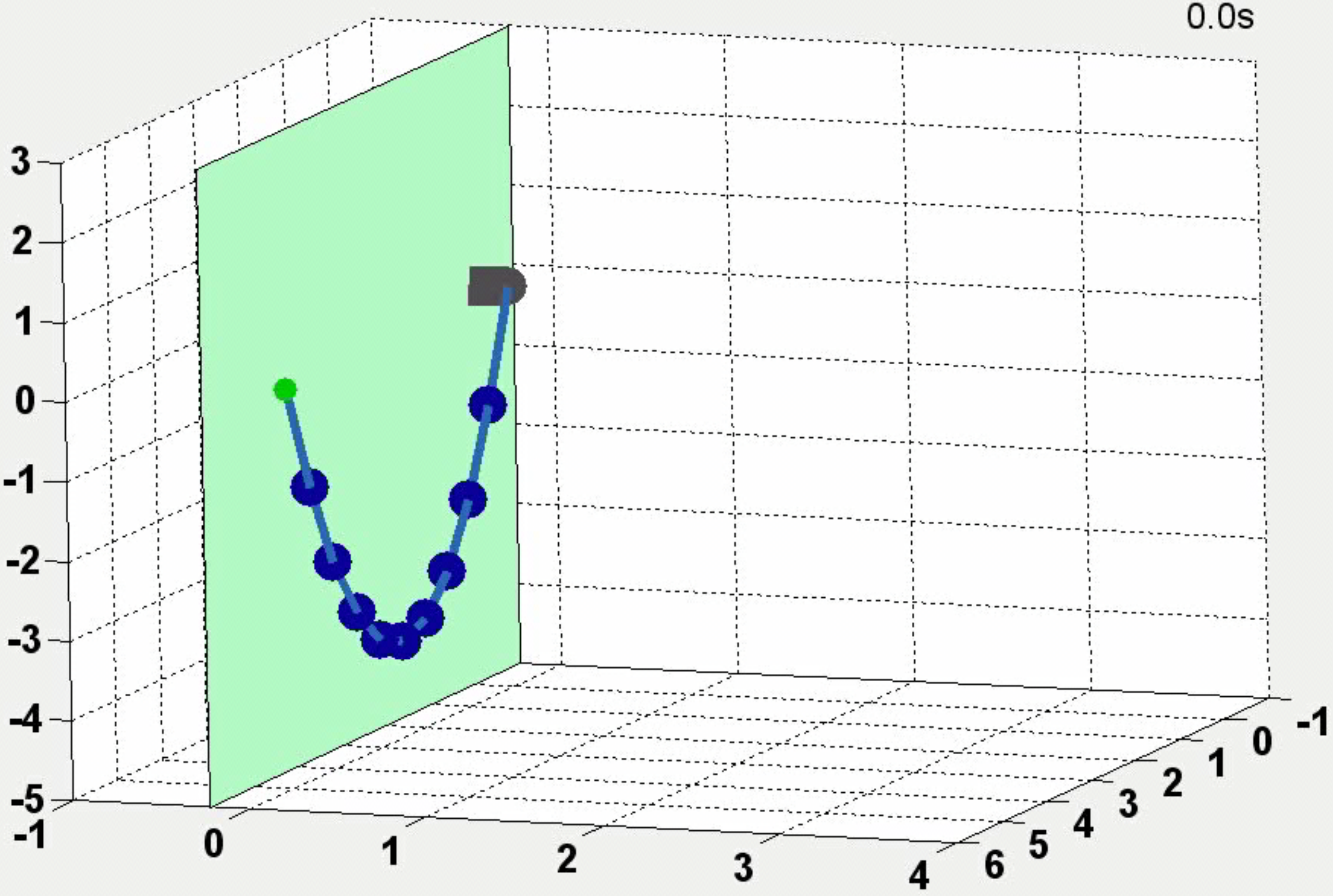
$$\ddot{x}_i = \frac{1}{m}\left(F_{i,i+1} - F_{i-1,i}\right) + g, \quad i = 1, \ldots, N$$

- Force according to Hooke's law strongly nonlinear:

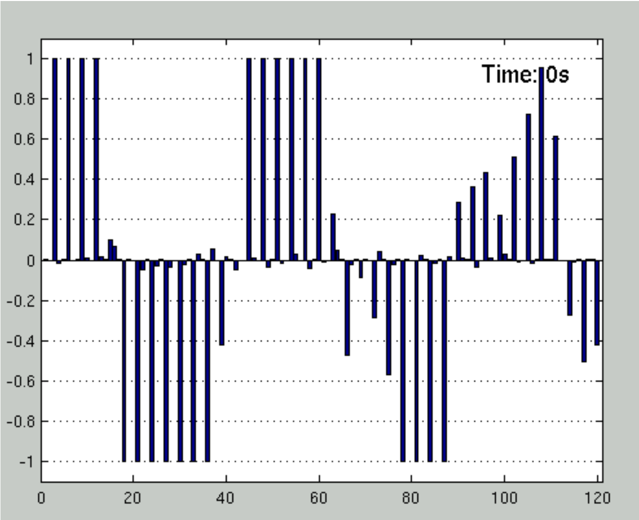
$$F_{i,i+1} = D\left(1 - \frac{L}{\|x_{i+1} - x_i\|}\right)(x_{i+1} - x_i),$$

- Together: 57 nonlinear ODEs, chaotic system

After disturbance, chain crashes into wall



MPC controller shall avoid crashing into wall

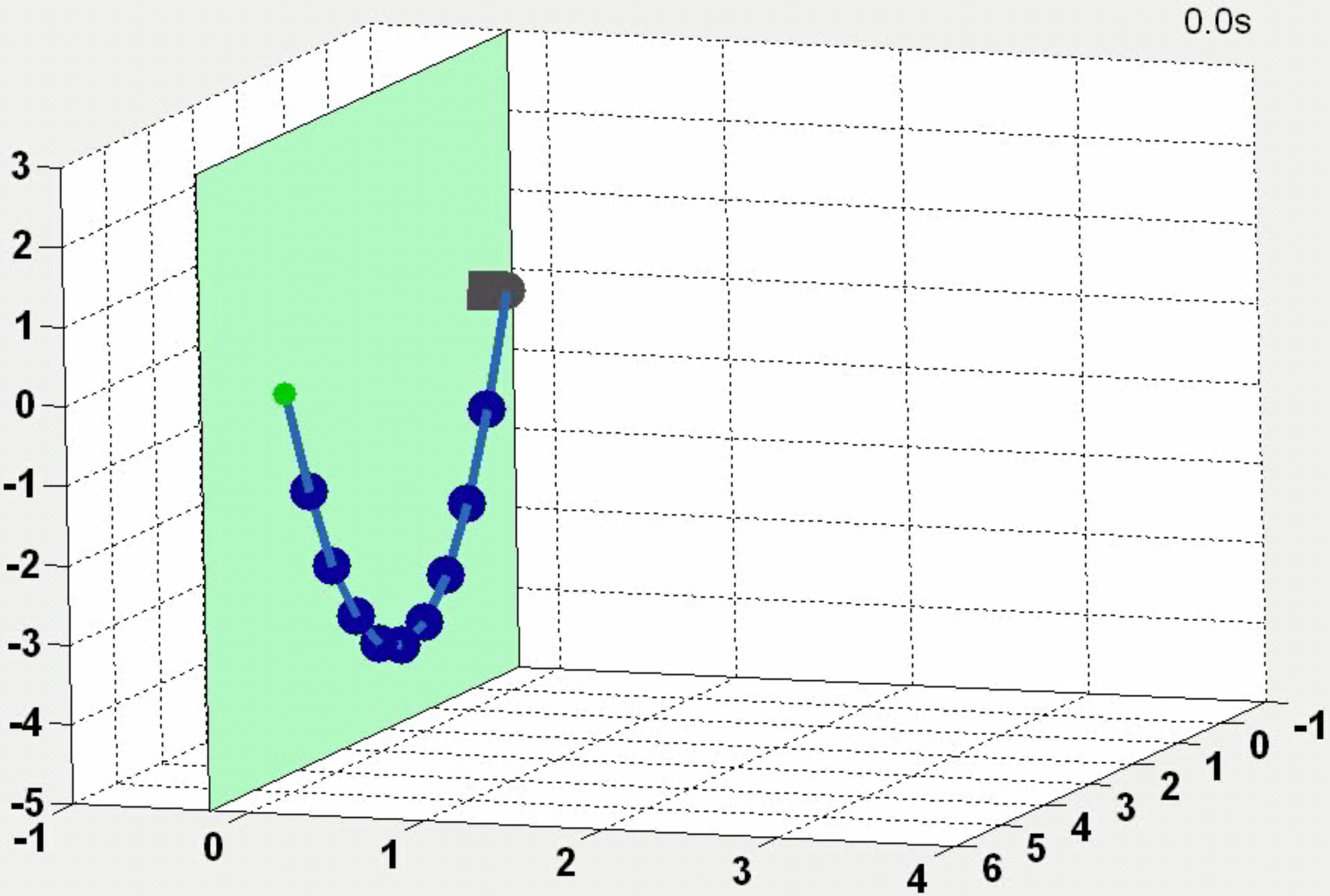


first half of MPC horizon

- linearize system at steady state
- choose 200 ms sampling time
- predict 80 samples:
  3 x 80 = **240 degrees of freedom**
- **bounds** (up/lo): 2 x 240 = **480**
- **state constraints** that avoid hitting the wall: 9 x 80 = **720**

Note: large QP with ~1 MB data

MPC respects bounds and state constraint

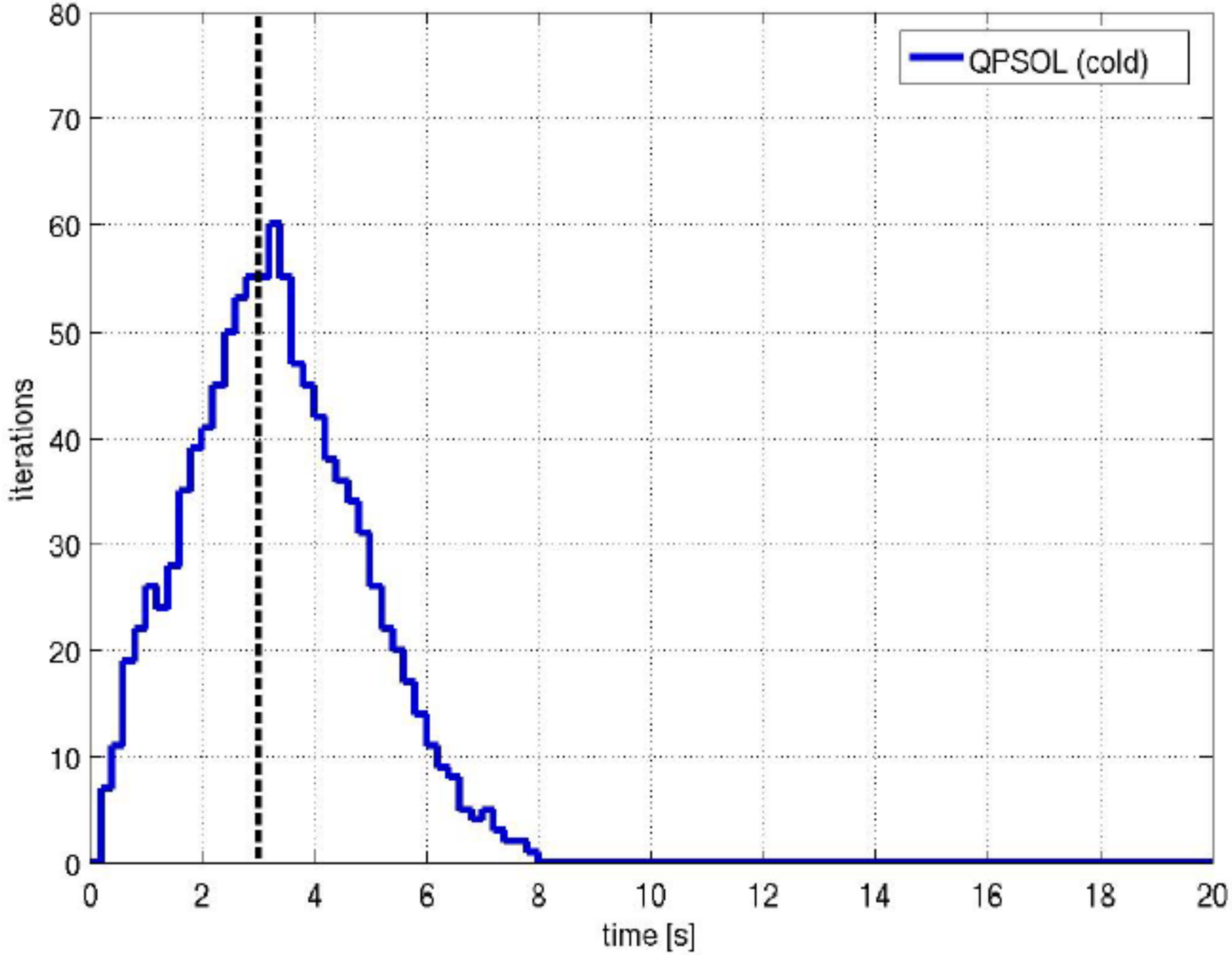


Compare four QP strategies

- Standard solver (QPSOL), cold start
- QPSOL, warm start
- Online Active Set Strategy (qpOASES), full convergence
- qpOASES, real-time variant with at most 10 QP Iterations

Note: Explicit MPC cannot be applied due to problem size

Number of QP Iterations (Working Set Changes)



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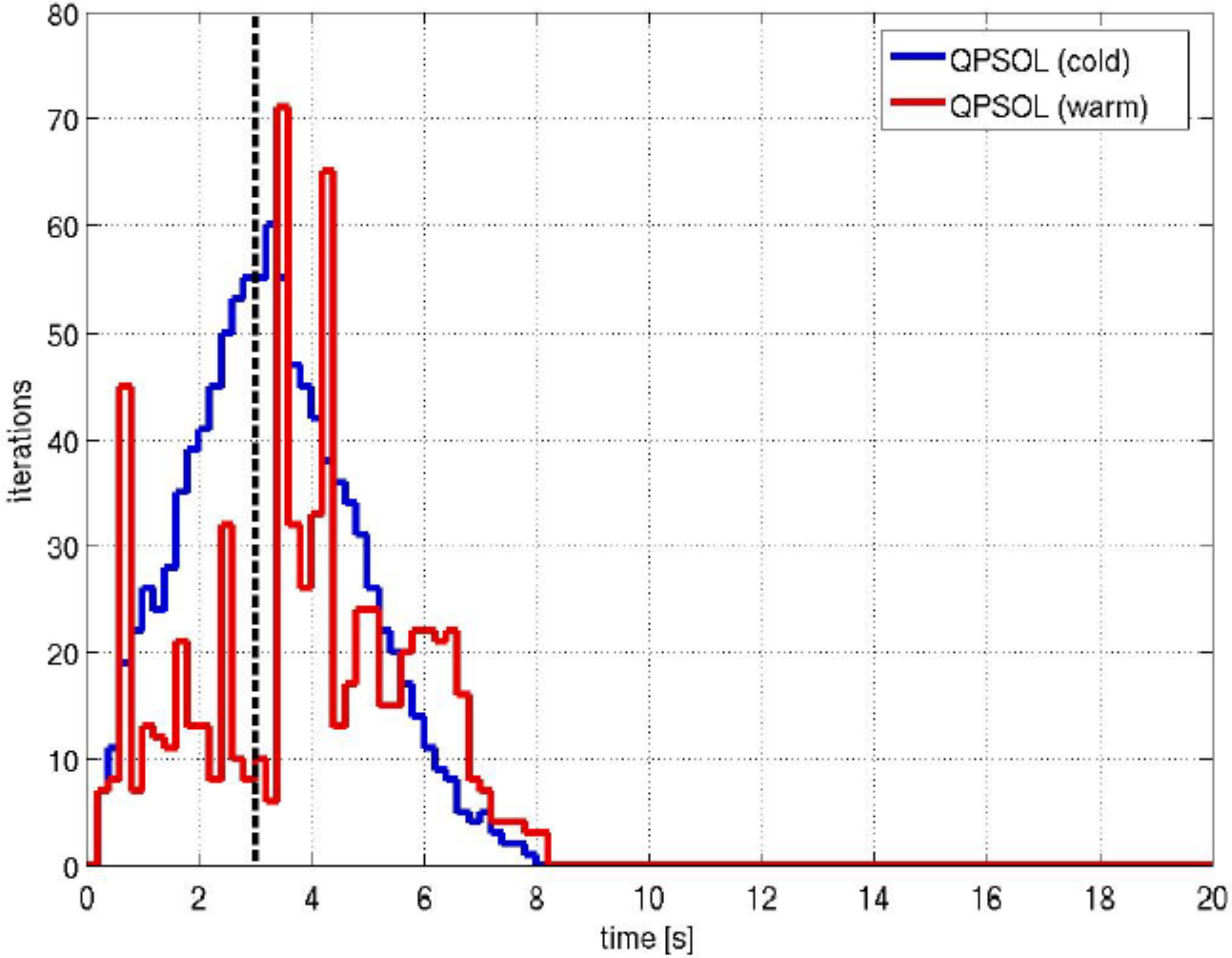

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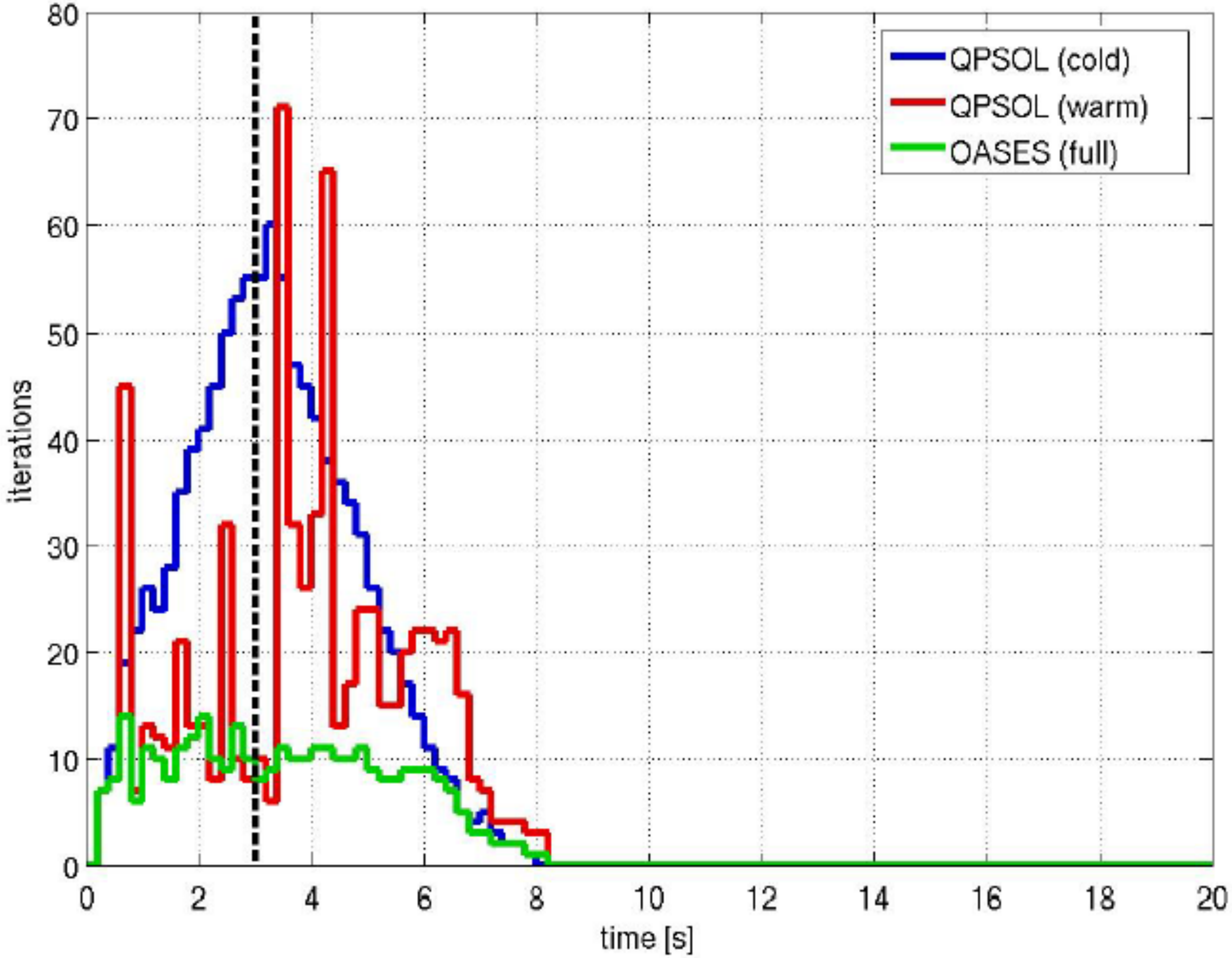

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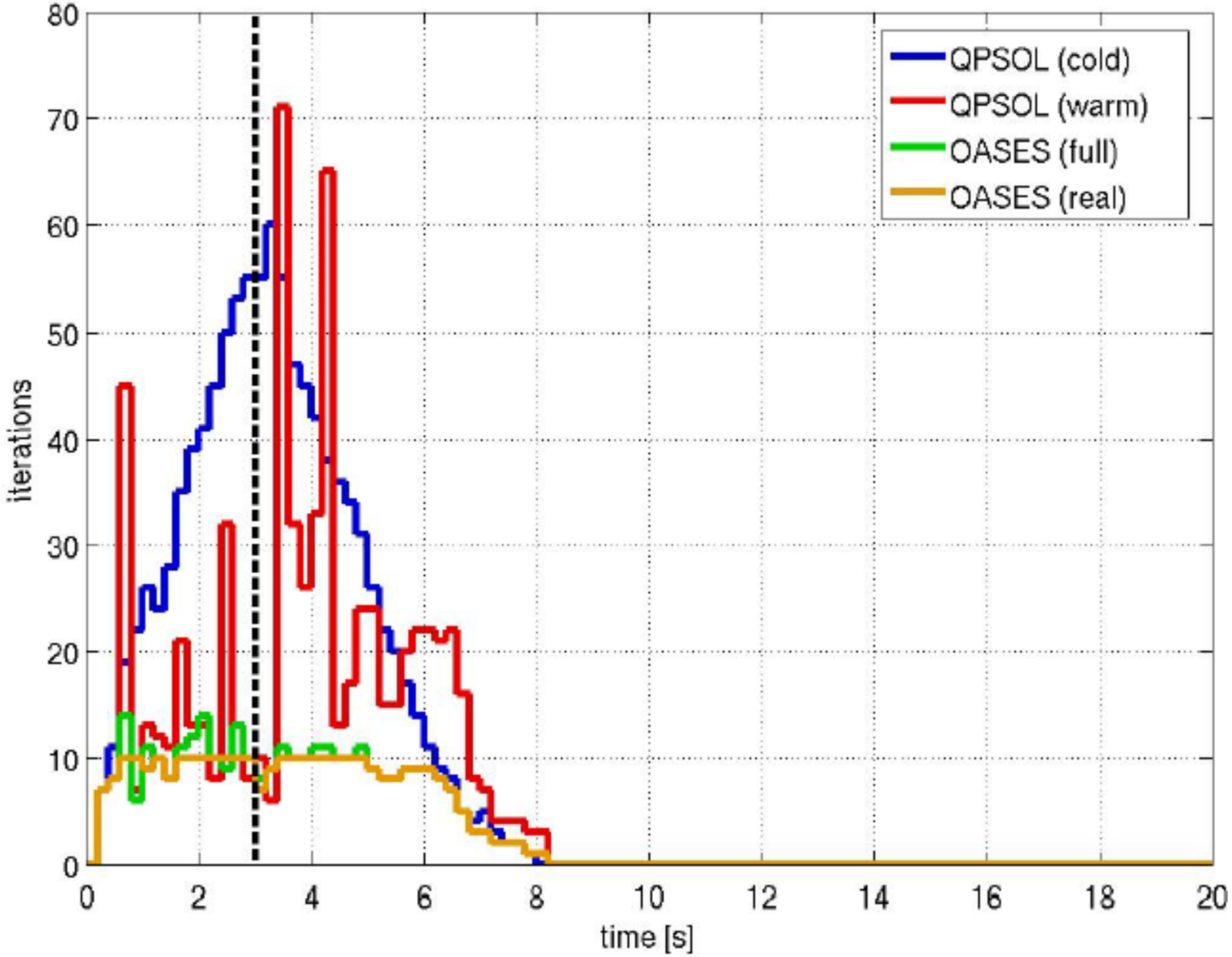


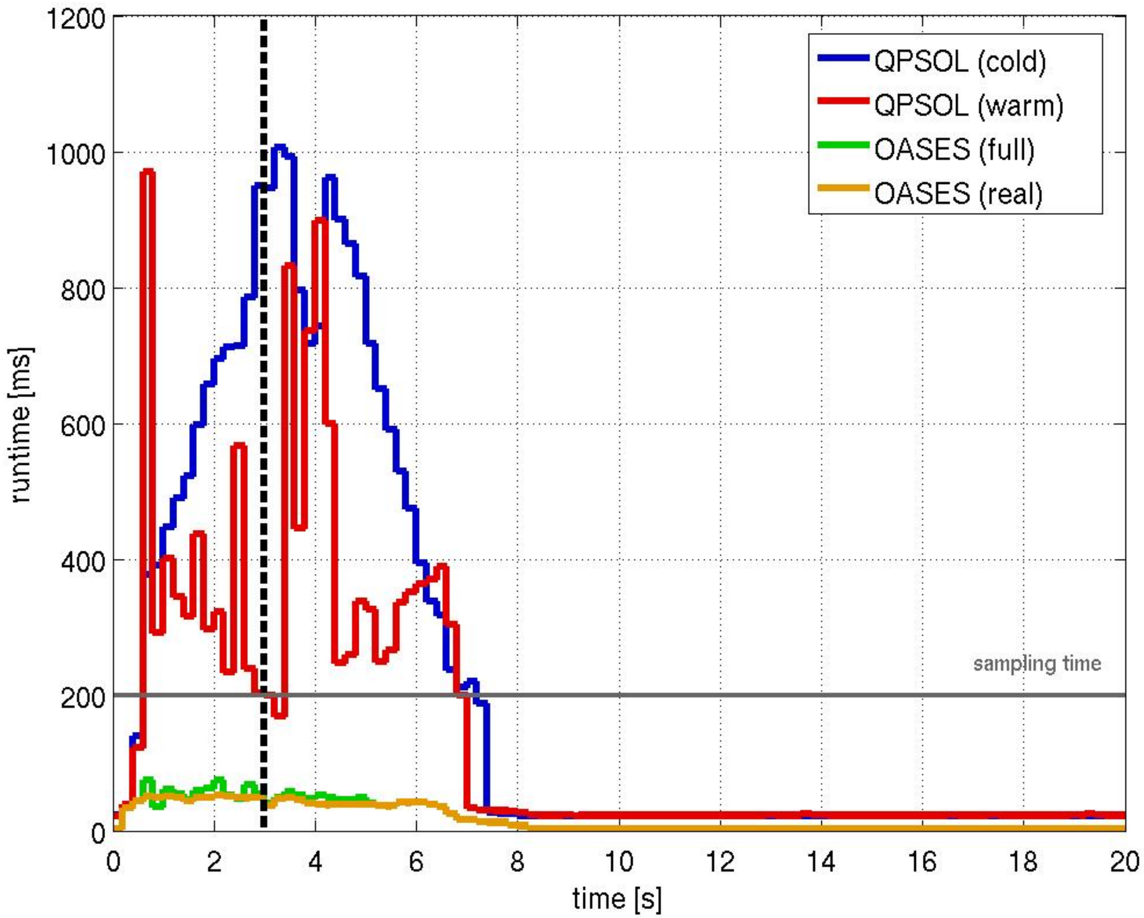
Performance of Online Active Set Strategy

- Number of QP working set changes 3-5 times lower than for QPSOL with warm starts
- Can limit maximum number without much suboptimality

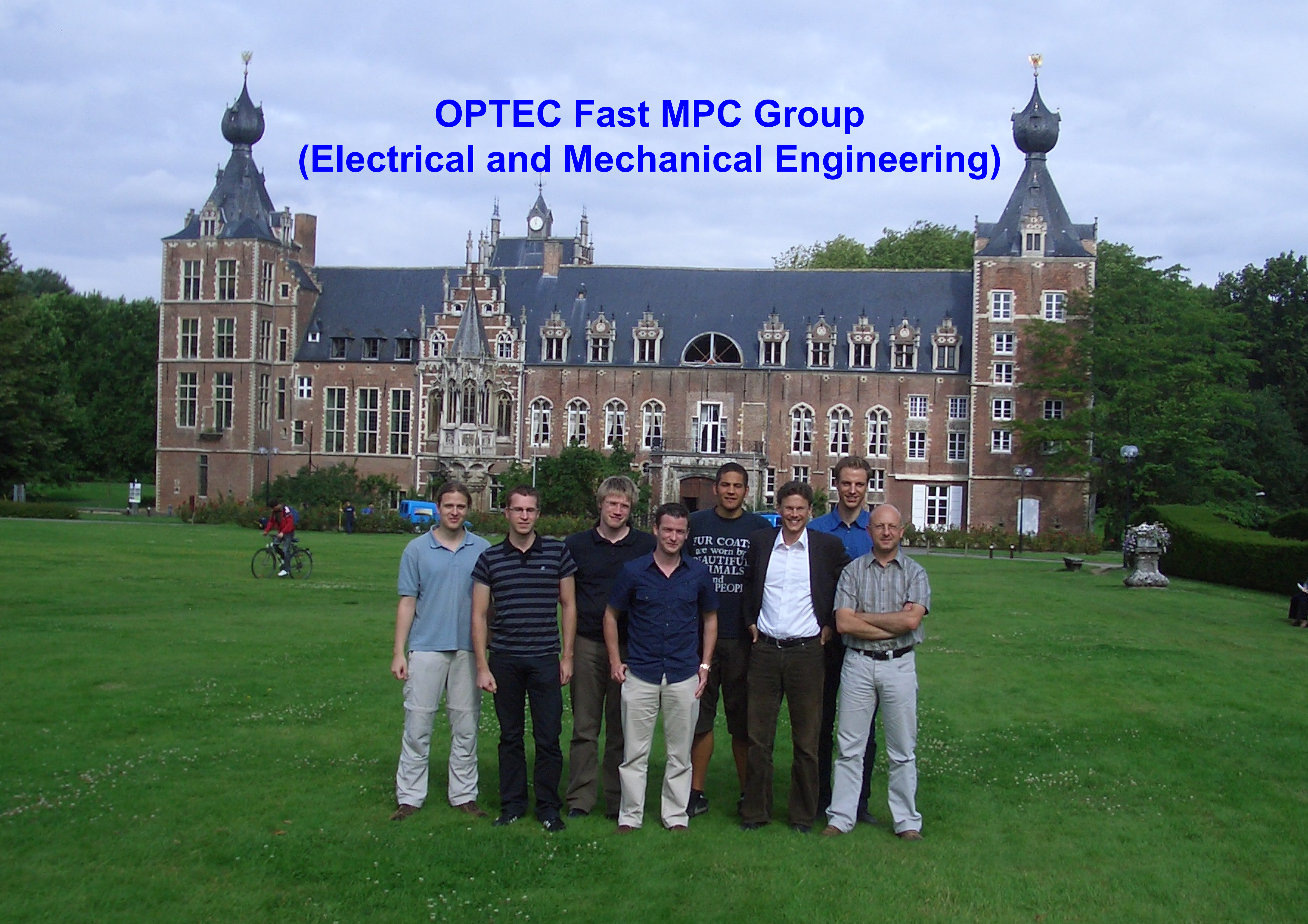
Additionally:

- QPSOL often even needs Phase1 LP Iterations
- qpOASES needs no new matrix factorizations
- CPU times compare even more favourably...

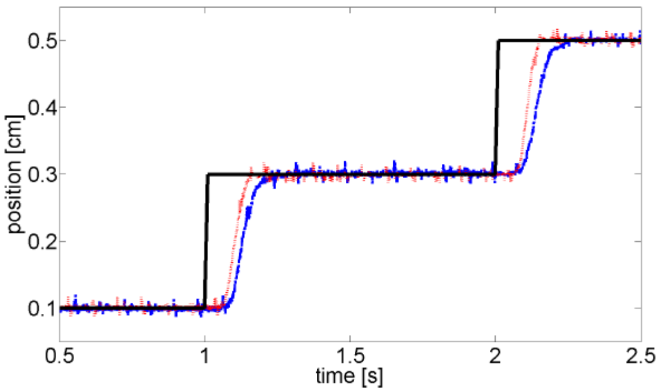
CPU Time Comparison: qpOASES Factor 10 Faster

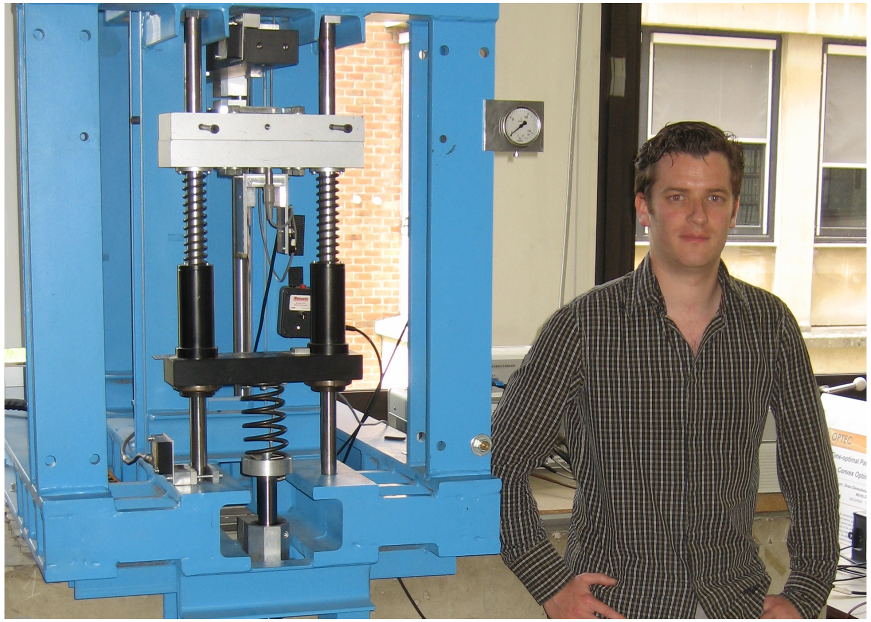


OPTEC Fast MPC Group
(Electrical and Mechanical Engineering)



Time Optimal MPC: a 100 Hz Application

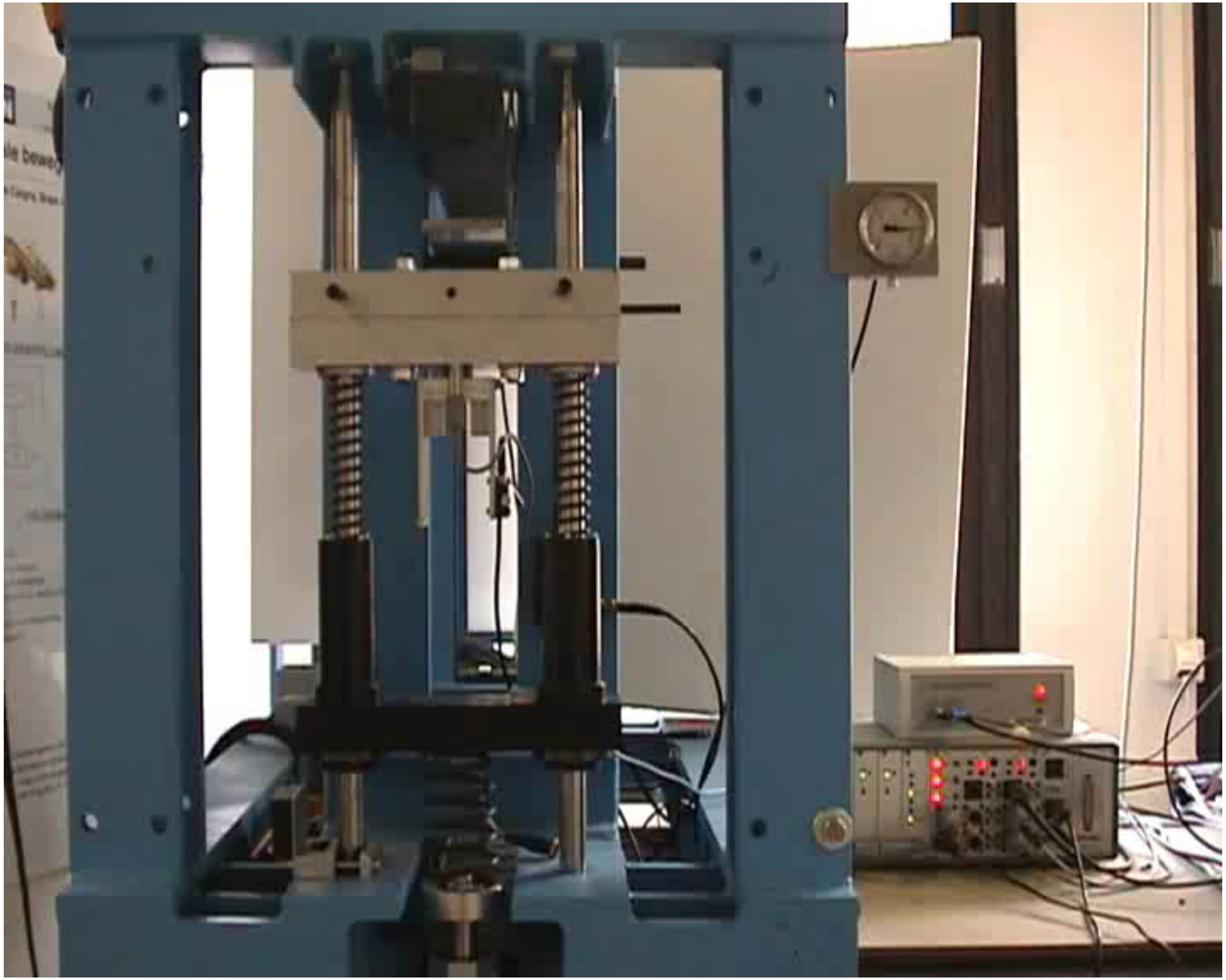




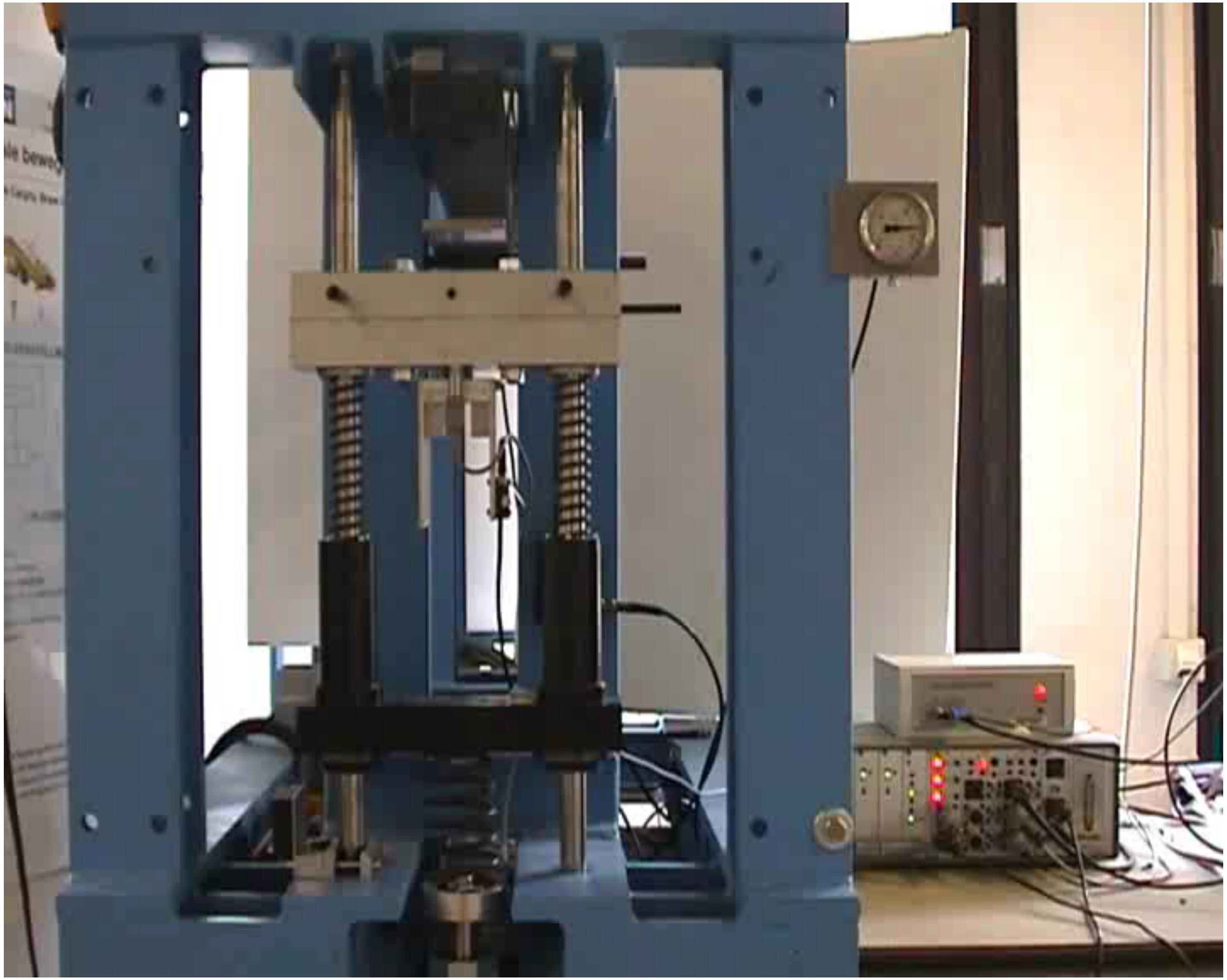
- Quarter car: oscillating spring damper system
- MPC Aim: settle at any new setpoint in ***in minimal time***
- Two level algorithm: MIQP
  - 6 online data
  - 40 variables + one integer
  - 242 constraints (in-&output)
- use qpOASES on dSPACE
- **CPU time: <10 ms**

Lieboud Van den Broeck in front of quarter car experiment

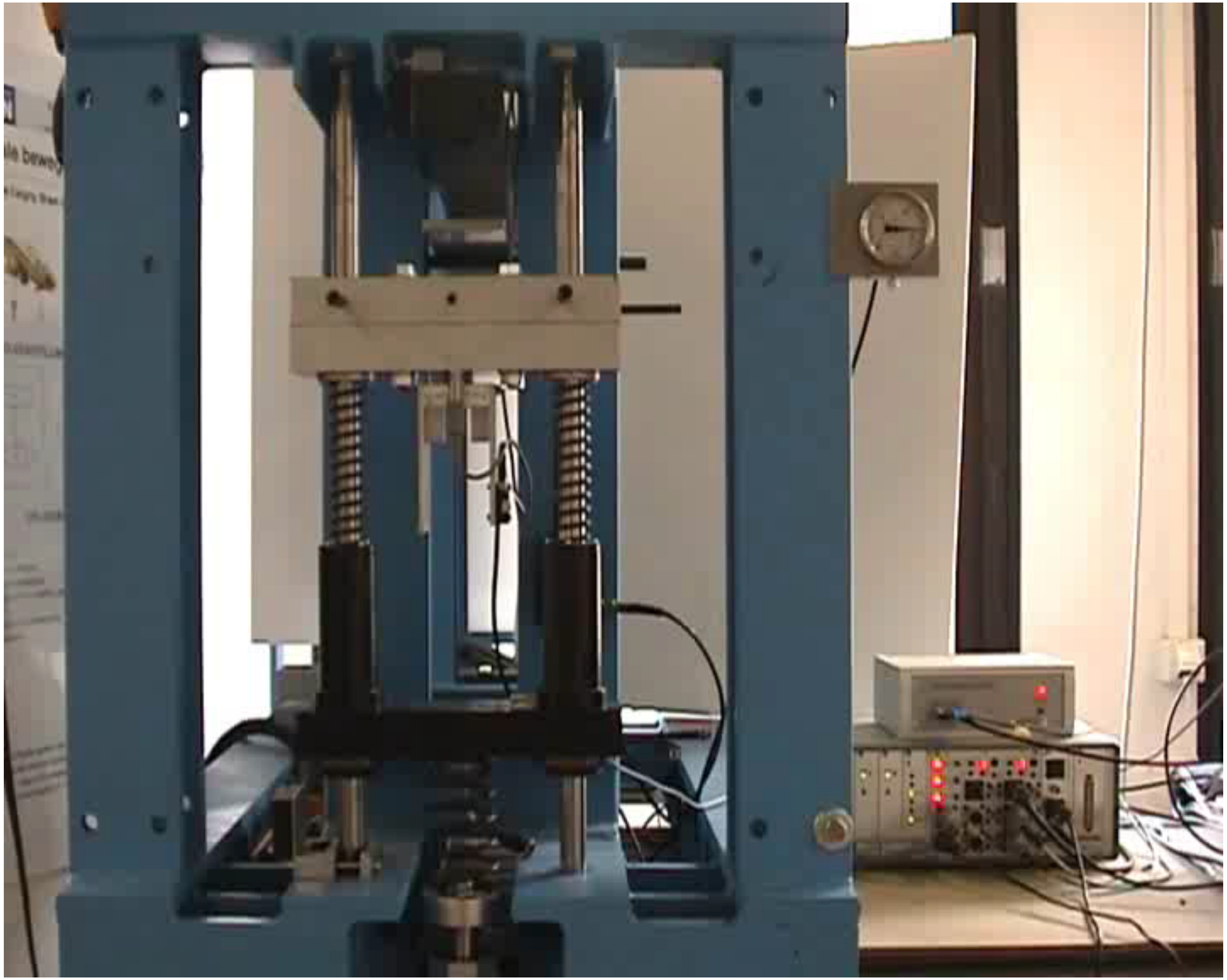
Setpoint change without control: oscillations



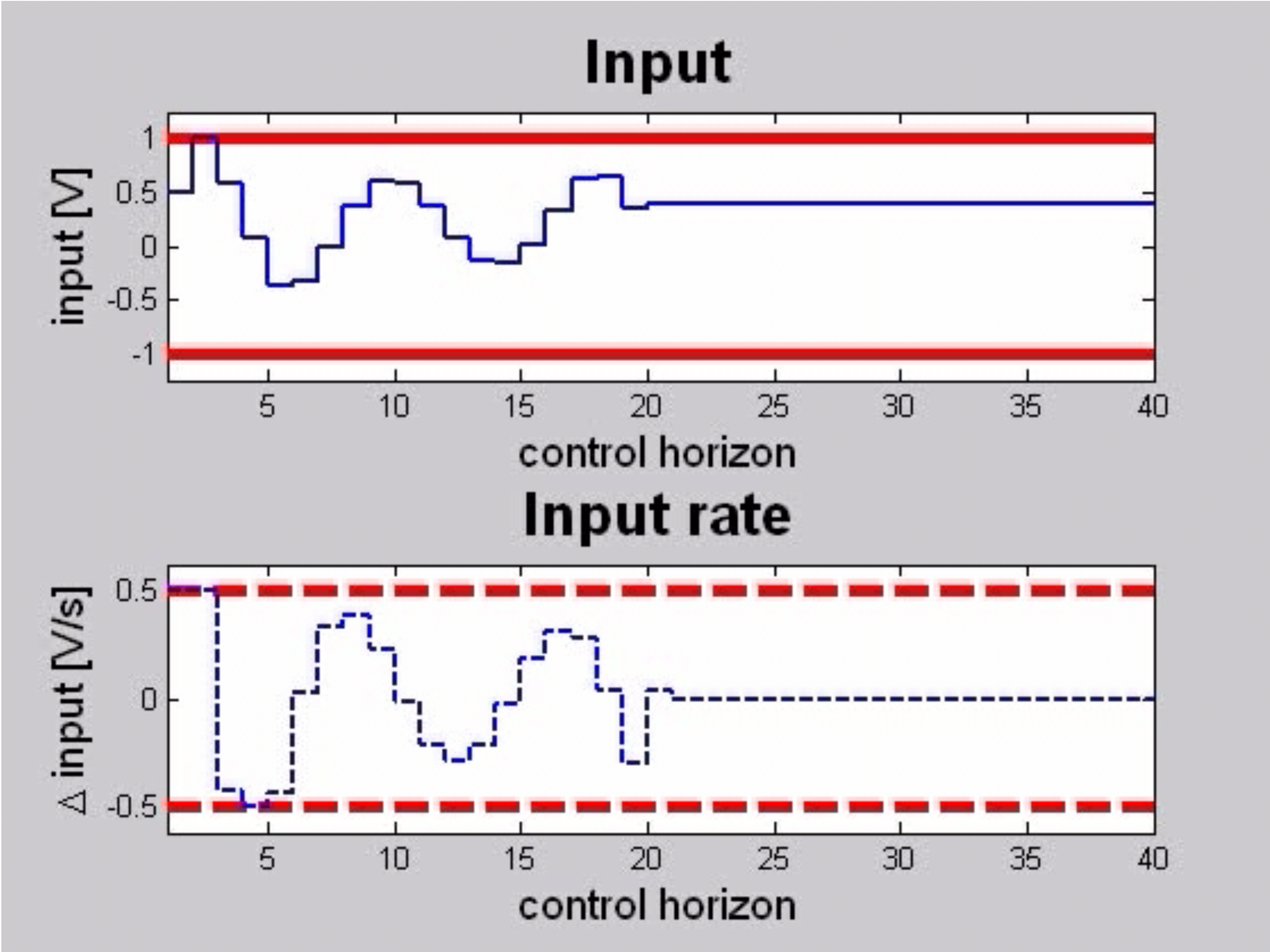
With LQR control: inequalities violated



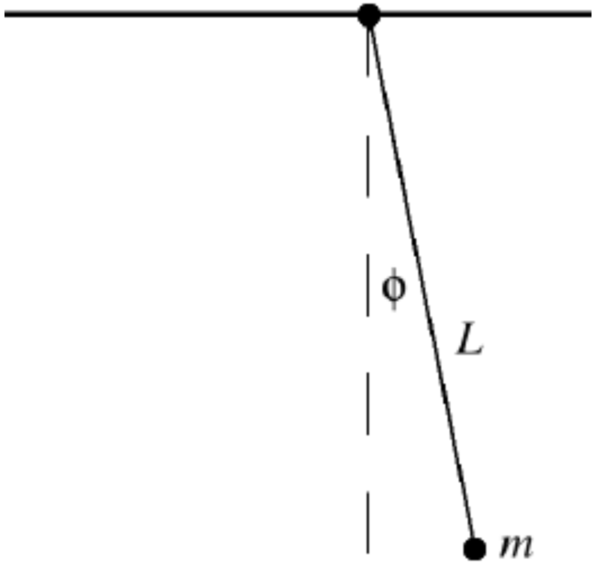
With Time Optimal MPC

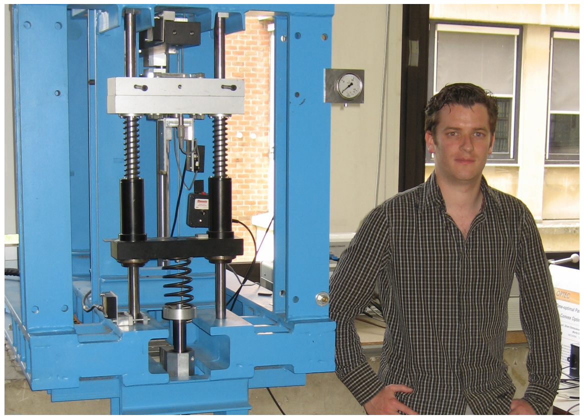


Time Optimal MPC: qpOASES Optimizer Contents



Time Optimal MPC: a 60 Hz Application





Lieboud Van den Broeck

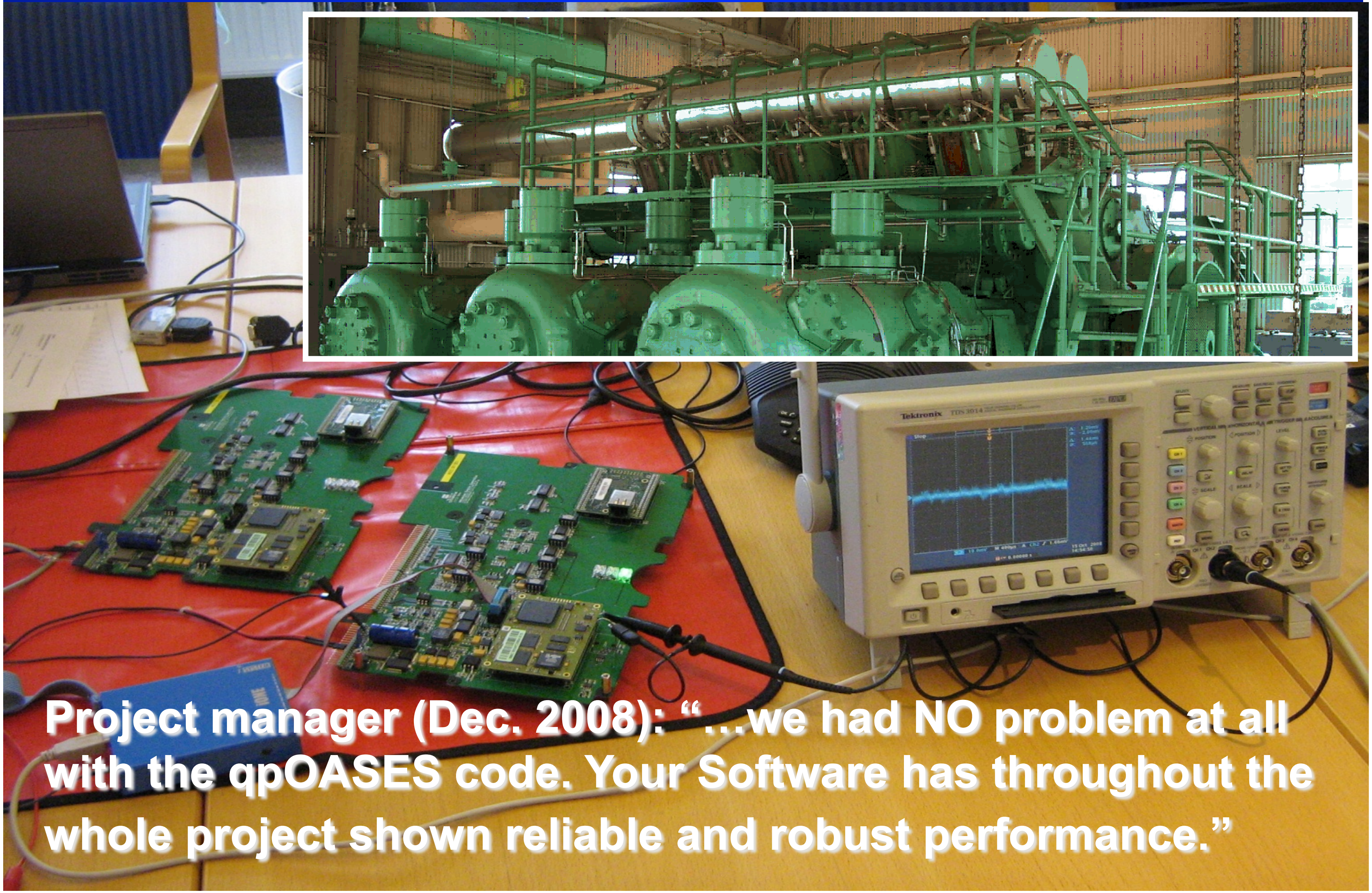
- Overhead crane
- MPC Aim: settle at any new setpoint in ***in minimal time***
- Two level algorithm: MIQP
  - 6 online data
  - 40 variables + one integer
  - 242 constraints (in-&output)
- use qpOASES on dSPACE
- **CPU time: <10 ms**

qpOASES: open code, but direct industrial funding

Hoerbiger: MPC of Large Bore Gas Engines for US Pipeline Compressors

IPCOS: Efficient QP for Process Control

qpOASES running on Industrial Control Hardware (20 ms)



Project manager (Dec. 2008): "…we had NO problem at all with the qpOASES code. Your Software has throughout the whole project shown reliable and robust performance."

Conclusions

- Linear and Nonlinear MPC need reliable QP solution
- Explicit MPC prohibitive for nontrivial problem dimensions
- Online Active Set Strategy (qpOASES) is one order of magnitude faster than conventional QP with warmstarts
- Linear MPC in kHz range realizable even for larger QPs
- Time Optimal MPC interesting alternative to tracking