the max-norm in a Banach space B of continuous vectorvalued functions over an interval $I = [a - d, a + d]$, i.e.,

$$\|u\| = \max_{t \in I} |u(t)|.$$

These notations are also used for the corresponding operator norms. Let $D \subseteq \mathbf{R}^s$ be a closed region. We recall, see Sec. 12.2, that f satisfies a Lipschitz condition in D, with the Lipschitz constant L, if

$$|f(y) - f(z)| \leq L|y - z|, \quad \forall y, z \in D. \tag{13.1.6}$$

By Lemma 11.2.2, $\max |f'(y)|$, $y \in D$, is a Lipschitz constant, if f is differentiable and D is convex. A point, where a local Lipschitz condition is not satisfied is called a **singular point** of the system (13.1.5).

THEOREM 13.1.1.

If f satisfies a Lipschitz condition in the whole of $\mathbf{R}^s$, then the initial value problem (13.1.5) has precisely one solution for each initial vector c. The solution has a continuous first derivative for all t.

If the Lipschitz condition holds in a subset D of $\mathbf{R}^s$ only, then existence and uniqueness holds as long as the orbit stays in D.

Proof: We shall sketch a proof of this fundamental theorem, when $D = \mathbf{R}^s$, based on an iterative construction named after Picard. We define an operator F (usually nonlinear) that maps the Banach space B into itself:

$$F(y)(t) = c + \int_a^t f(y(x))dx.$$

Note that the equation $y = F(y)$ is equivalent to the initial value problem (13.1.5) on some interval $[a - d, a + d]$, and consider the iteration, $y_0 = c$ (for example),

$$y_{n+1} = F(y_n).$$

For any pair y, z of elements in B, we have,

$$\begin{aligned}\|F(y) - F(z)\| &\leq \int_a^{a+d} |f(y(t)) - f(z(t))| \cdot |dt| \\ &\leq \int_a^{a+d} L|y(t) - z(t)| \cdot |dt| \leq Ld\|y - z\|.\end{aligned}$$

It follows that Ld is a Lipschitz constant of the operator F. If $d < 1/L$, F is a contraction, and it follows from the Contraction Mapping (Theorem 11.2.1) that the equation $y = F(y)$ has a unique solution. For the initial value problem (13.1.5) it follows that there exists precisely one solution, as long as $|t - a| \leq d$. This solution can then be continued to any time by a step by step procedure, for $a + d$ can be chosen as a new starting time and substituted for a in the proof. In this way we extend the solution to $a + 2d$, then to $a + 3d$, $a + 4d$ etc. (or backwards to $a - 2d$, $a - 3d$, etc.). ∎

Note that this proof is based on two ideas of great importance to numerical analysis: *iteration* and the *step-by-step construction*. (There is an alternative proof that avoids the step-by-step construction, see, e.g., Coddington and Levinson, [2, 1955, p. 12]). A few points to note are:

A. For the *existence* of a solution, it is *sufficient that f is continuous*, (the existence theorem of Cauchy and Peano, see, e.g., Coddington and Levinson [2, 1955, p.6])). That *continuity is not sufficient for uniqueness* can be seen by the following simple initial value problem,

$$y' = 2|y|^{1/2}, \qquad y(0) = 0,$$

which has an infinity of solutions for $t > 0$, namely $y(t) = 0$, or, for any non-negative number k,

$$y(t) = \begin{cases} 0, & \text{if } t \le k; \\ (t-k)^2, & \text{otherwise.} \end{cases}$$

B. The theorem is *extended to non-autonomous systems* by the usual device for making a non-autonomous system autonomous (see Sec. 13.1.1).

C. If the Lipschitz condition holds only in a subset D, then the ideas of the proof can be extended to guarantee existence and uniqueness, *as long as the orbit stays in D*. Let M be an upper bound of $|f(y)|$ in D, and let r be the shortest distance from c to the boundary of D. Since

$$|y(t) - c| = |\int_a^t f(y(x))dx| \le M|t-a|,$$

we see that there will be no trouble as long as $|t-a| < r/M$, at least. (This is usually a pessimistic underestimate.) On the other hand, the example

$$y' = y^2, \qquad y(0) = c > 0,$$

which has the solution $y(t) = c/(1-ct)$, shows that the solution can cease to exist for a finite t (namely for $t = 1/c$), even if $f(y)$ is differentiable for all y. Since $f'(y) = 2y$, the Lipschitz condition is guaranteed only as long as $2y < L$. In this example, such a condition cannot hold forever, no matter how large L has been chosen.

D. On the other side: the solution of a *linear* non-autonomous system, where the data (i.e. the coefficient matrix and the right hand side) are *analytic* functions in some domain of the complex plane, cannot have other singular points than the data, in the sense of complex analysis.

E. Isolated *jump discontinuities* in the function f offer no difficulties, if the problem after a discontinuity can be considered as a new initial value problem that satisfies a Lipschitz condition. For example, *in a non-autonomous problem of the form*

$$y' = f(y) + r(t),$$

existence an uniqueness holds, even if the driving function $r(t)$ is only piecewise continuous. In this case $y'(t)$ is discontinuous, only when $r(t)$ is so, hence $y(t)$ is continuous. There exist, however, more nasty discontinuities, where existence and uniqueness are not obvious, see Problem 3.

F. A point y^* where $f(y^*) = 0$ is called a **critical point** of the autonomous system. (It is usually not a singular point.) If $y(t_1) = y^*$ at some time t_1, the theorem tells that $y(t) = y^*$ is the only solution for all t, forwards as well as backwards. It follows that a solution that does not start at y^* cannot reach y^* exactly in finite time, but it can converge very fast towards y^*.

Note that this does not hold for a non-autonomous system, at a point where $f(t_1, y(t_1)) = 0$, as is shown by the simple example $y' = t$, $y(0) = 0$, for which $y(t) = \frac{1}{2}t^2 \neq 0$ when $t \neq 0$. *For a non-autonomous system $y' = f(t, y)$, a critical point is instead defined as a point y^*, such that $f(t, y^*) = 0$, $\forall t \geq a$.* Then it is true that $y(t) = y^*, \forall t \geq a$, if $y(a) = y^*$.

13.1.3. Variational Equations and Error Propagation We first discuss the propagation of disturbances (for example numerical errors) in an ODE system. It is a useful model for the error propagation in the application of one step methods, i.e. if y_n is the only input data to the step, where y_{n+1} is computed.

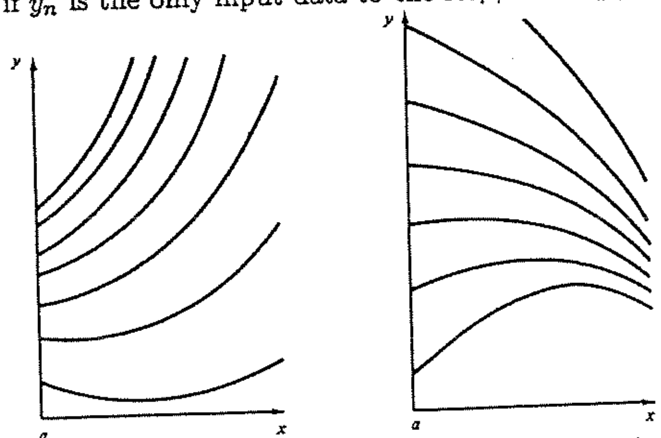

FIG. 13.1.2. *Two families of solution curves $y(t;c)$.*

The solution of the initial-value problem, (13.1.3), can be considered as a function $y(t; c)$, where c is the vector of initial conditions. Here again, one can visualize a *family of solution curves*, this time in the (t, y)-space, one curve for each initial value, $y(a; c) = c$. For the case $s = 1$, the family of solutions can, for example, look one of the two set of curves in Fig. 13.1.2a,b. The dependence