Using quantitative diagnosability analysis for optimal sensor placement

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Abstract: A good placement of sensors is crucial to get good performance in detecting and isolating faults. Here, the sensor placement problem is cast as a minimal cost optimization problem. Previous works have considered this problem with *qualitative* detectability and isolability specifications. A key contribution here is that *quantified* detectability and isolability performance is considered in the optimization formulation. The search space for the posed optimization problem is exponential in size, and to handle complexity a greedy optimization algorithm that compute optimal sensor positions is proposed. Two examples illustrate how the optimal solution depends on the required quantified diagnosability performance and the results are compared to the solutions using a deterministic method.

Keywords: Fault isolation, Diagnosis, Sensor placement

1. INTRODUCTION

In model-based diagnosis, diagnosis is performed by comparing on-line system information and a system model. The on-line information is mostly obtained by installed sensors and therefore the placement of the sensors is important for the diagnosability performance.

Sensor placement for fault diagnosis has been treated in several papers. Example of previous works are Yassine et al. (2008), Commault and Dion (2007), Krysander and Frisk (2008), Raghuraj et al. (1999), and Trave-Massuyes et al. (2006) which all use structural descriptions of the model to find a set of sensors which achieves a required deterministic isolability performance. Deterministic isolability states whether a fault is isolable or not, given the selected set of sensors. In Rosich et al. (2010) and Debouk et al. (2002) the optimal minimum cost sensor set is sought given a required deterministic isolability performance. In Frisk et al. (2009) an analytical approach is used to find all sets of sensors fulfilling the required deterministic isolability performance.

A limitation of deterministic isolability analyses is that they only provide a yes or no answer to questions like: is a fault detectable? Sensor placement based on deterministic isolability can provide sensor sets that in practice are not good for diagnosis due to noise and model uncertainties.

A method for analyzing quantified diagnosability performance, *distinguishability*, was introduced in Eriksson et al. (2011b) for linear static models, and extended to time-discrete dynamic linear descriptor models in Eriksson et al. (2011a). Distinguishability is used in this paper to optimize sensor placement for fault diagnosis to find a cheapest sensor set which achieves a required quantified diagnosability performance. The proposed method is applied to two example models where the solutions are analyzed and compared to the results using a deterministic method.

2. INTRODUCTORY EXAMPLE

Before presenting the problem formulation in this paper, the result of using a deterministic algorithm on a linear model for finding optimal sensor sets will be discussed. Then a discussion will follow on how the performance of a diagnosis algorithm, based the computed set of sensors, is affected by model uncertainties and why this should be considered when finding optimal sensor sets.

A discretized version of a small continuous linear dynamic example model, discussed in Krysander and Frisk (2008),

$$\begin{aligned} x_1[t+1] &= x_2[t] + x_5[t] \\ x_2[t+1] &= -x_2[t] + x_3[t] + x_4[t] \\ x_3[t+1] &= -2x_3[t] + x_5[t] + f_1[t] + f_2[t] \\ x_4[t+1] &= -3x_4[t] + x_5[t] + f_3[t] \\ x_5[t+1] &= -4x_5[t] + f_4[t] \end{aligned} \tag{1}$$

is considered where x_i are state variables and f_i are modeled faults.

A deterministic method finds sets of sensors that achieves maximum deterministic fault isolability, i.e., a set of sensors which makes it possible to isolate all faults that are isolable from each other. A set of sensors which fulfills maximum deterministic fault isolability, where no subset of sensors fulfills it, is called a *minimal sensor set*, see Krysander and Frisk (2008).

2.1 Sensor placement using deterministic method

If x_i in (1) are possible sensor locations and model uncertainties and measurement noise are ignored, a deterministic analysis of maximum fault isolation can be performed, e.g., using the method in Frisk et al. (2009). Maximum deterministic fault isolability can be computed by including all possible sensors, and the result is summarized in Table 1. An X in position i, j represents that fault mode f_i is isolable from f_j and a 0 if not. The NF column represents if the fault is detectable, i.e., if f_i is isolable from the no fault case then it is detectable. The analysis shows that all faults are detectable, f_1 and f_2 are isolable from the faults f_3 and f_4 but not from each other, and that f_3 and f_4 are fully isolable from the other faults.

It is not necessary to measure all states x_i in (1) to achieve the isolability in Table 1. Applying the deterministic sensor placement method in Krysander and Frisk (2008), gives all minimal sensor sets,

$$\{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \text{ and } \{x_3, x_4\}, \tag{2}$$

that achieve the deterministic fault isolability in Table 1. Each set in (2), and all supersets, represents all sensor sets that achieves maximum deterministic fault isolability.

The minimal sensor sets in (2) are found without taking model uncertainties and measurement noise into consideration. If model uncertainties and measurement noise are considered, the choice of minimal sensor set will affect the achieved diagnosability performance. The deterministic analysis does not state which sensor set in (2) that will give the best performance of a diagnosis system. It neither gives any information if the number of sensors is enough to get sufficient diagnosability performance.

2.2 Analysis of minimal sensor sets using distinguishability

In Eriksson et al. (2011b) and Eriksson et al. (2011a) a measure, distinguishability, for quantifying diagnosability performance of time-discrete linear dynamic systems was introduced. Distinguishability gives the upper limit for the fault to noise ratio, FNR, of an residual by considering model uncertainties and fault time profiles. A *fault time profile* is a vector $\theta_i = (\theta[t-n+1], \ldots, \theta[t])^T$ describing how the fault, $f_i = \theta_i$, varies during a time period of length n. A higher distinguishability value corresponds to a higher diagnosability performance.

Before computing distinguishability for the different minimal sensor sets, (2), some assumptions are made. First, all possible sensors in this example are assumed to have additive measurement noise which is i.i.d. Gaussian with variance one, i.e., $y_i = x_i + e_i$ where $e_i \sim \mathcal{N}(0,1)$. For simplicity, it is assumed that the added sensors can not become faulty, i.e., no new faults are introduced in the model. It is also assumed that the system is observed for a time window length of five samples, and that the faults to be isolated are constant faults with amplitude one, i.e. $\theta_i = \bar{1} = (1, 1, \ldots, 1)^T$ for each fault mode f_i.

Consider first the minimal sensor set $\{x_2, x_3\}$ in (2). The computed distinguishability is presented in Table 2. A nonzero value in position i, j corresponds to a constant fault f_i is isolable from the fault mode f_j. A higher distinguishability value means that the fault is easier to detect or isolate. The same information about deterministic isolability performance as in Table 1 can be stated in Table 2 since all non-isolable fault pairs have distinguishability value zero. Table 2 also shows that, for example, it is easier to detect f_1 than f_3, since 0.308 is greater than 0.033, and it is easier to isolate f_1 from f_3 than vice versa since 0.230 is greater than 0.020.

If instead the minimal sensor set $\{x_2, x_4\}$ in (2) is used, the computed distinguishability is presented in Table 3. A comparison of Table 2 and Table 3 gives that the sensor set $\{x_2, x_4\}$ makes it easier to detect and isolate f_3 from f_1 than the sensor set $\{x_2, x_3\}$, since 0.123 is greater than 0.020, but more difficult to detect and isolate f_1 and f_3, since 0.037 is less than 0.230.

Table 1. Achievable maximum fault isolability of the example model (1).

	NF	f_1	f_2	f_3	f_4
f_1	X	0	0	X	X
f_2	X	0	0	X	X
f_3	X	X	X	0	X
f_4	X	X	X	X	0

Table 2. Distinguishability for each fault pair $\{f_i, f_j\}$, if $f_i = \bar{1}$, given the sensor set $\{x_2, x_3\}$.

$\{x_2, x_3\}$	NF	f_1	f_2	f_3	f_4
f_1	0.308	0	0	0.230	0.017
f_2	0.308	0	0	0.230	0.017
f_3	0.033	0.020	0.020	0	0.017
f_4	0.018	0.001	0.001	0.010	0

Table 3. Distinguishability for each fault pair $\{f_i, f_j\}$, if $f_i = \bar{1}$, given the sensor set $\{x_2, x_4\}$.

$\{x_2, x_4\}$	NF	f_1	f_2	f_3	f_4
f_1	0.062	0	0	0.037	0.023
f_2	0.062	0	0	0.037	0.023
f_3	0.171	0.123	0.123	0	0.023
f_4	0.014	0.005	0.005	0.002	0

The analysis shows that no minimal sensor set in (2) gives the best diagnosability performance for all pairs of fault modes. It could also be that none of the minimal sensor sets are sufficient to get satisfactory diagnosability performance in practice. If model uncertainties are considered, when finding an optimal sensor set, then the solution could be different from the solution of the deterministic analysis.

The example shows that if model uncertainties and measurement noise are not considered, when selecting a minimal sensor set, then sufficient diagnosability performance might not be achievable if the faults are too small. If process noise and measurement noise were considered then an optimal sensor set could be found which gives a required performance, for example FNR, of the diagnosis system.

3. PROBLEM FORMULATION

The objective here is to utilize *distinguishability* for quantified diagnosability performance to optimize sensor placement for fault diagnosis purposes. The type of models that will be considered are time-discrete linear dynamic descriptor models written as

$$\begin{aligned} Ex[t+1] &= Ax[t] + B_u u[t] + B_f f[t] + B_v v[t] \\ y[t] &= Cx[t] + D_u u[t] + D_f f[t] + D_\varepsilon \varepsilon[t] \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^{l_x}$ are state variables, $y \in \mathbb{R}^{l_y}$ are measured signals, $u \in \mathbb{R}^{l_u}$ are input signals, $f \in \mathbb{R}^{l_f}$ are modeled faults, $v \sim \mathcal{N}(0, \Lambda_v)$ and $\varepsilon \sim \mathcal{N}(0, \Lambda_\varepsilon)$ are i.i.d. Gaussian vectors with zero mean and symmetric positive definite covariance matrices $\Lambda_v \in \mathbb{R}^{l_v \times l_v}$ and $\Lambda_\varepsilon \in \mathbb{R}^{l_\varepsilon \times l_\varepsilon}$. The model matrices are of appropriate dimensions. Note that the matrix E can be singular.

Assume that a model (3), denoted with $\mathcal{M}$, and a set of possible sensors $\mathcal{O}$ are given. Each sensor $s \in \mathcal{O}$ has a sensor position and a known noise variance. Let $\mathcal{D}^{\mathcal{S}}_{i,j}(\theta_i; n)$ define distinguishability for a fault f_i with a given fault time profile θ_i and a window length n from a fault mode f_j for a given sensor set $\mathcal{S}$. A formal definition of distinguishability will be presented in Section 4. The objective is to find a minimum cost sensor set which fulfills a *minimum required distinguishability*, $\mathcal{D}^{\text{req}}_{i,j}(\theta_i; n)$, for each fault pair $\{f_i, f_j\}$.

The sensor placement problem is now formulated as an optimization problem,

$$\begin{aligned} &\min_{\mathcal{S} \subseteq \mathcal{O}} \; h(\mathcal{S}) \\ &\text{s.t. } \mathcal{D}^{\mathcal{S}}_{i,j}(\theta_i; n) \geq \mathcal{D}^{\text{req}}_{i,j}(\theta_i; n), \; \forall i, j, \end{aligned} \tag{4}$$

where $\mathcal{S} \subseteq \mathcal{O}$ is a set of selected sensors, $h(\mathcal{S})$ is a cost function, and $\mathcal{D}^{\mathcal{S}}_{i,j}(\theta_i; n)$ is the achieved distinguishability for each fault pair $\{f_i, f_j\}$ given the sensors $\mathcal{S}$. The cost function $h(s)$ could, for example, be the total sensor cost

$$h(s) = \sum_{s_l \in S} \text{cost}(s_l)$$

or the total number of sensors if $\text{cost}(s_l) = 1$ for all $s_l \in \mathcal{O}$.

The objective in this paper is, given a model $\mathcal{M}$ in the form (3) and an available set of sensors $\mathcal{O}$, to find a solution to (4). That is, finding a minimum cost sensor set which fulfills the required diagnosability performance defined by $\mathcal{D}^{\text{req}}_{i,j}(\theta_i; n)$.

4. BACKGROUND THEORY

The theory presented here is needed to define distinguishability. A more thorough description can be found in Eriksson et al. (2011b) and Eriksson et al. (2011a).

4.1 Model

Before analyzing the time-discrete descriptor model (3) it is written as a sliding window model, i.e., a sliding window of length n is applied to (3), see, e.g., Eriksson et al. (2011a). Define the vectors

$$\begin{aligned} z &= (y[t-n+1]^T, \ldots, y[t]^T, u[t-n+1]^T, \ldots, u[t]^T)^T \\ x &= (x[t-n+1]^T, \ldots, x[t+1]^T)^T, \\ f &= (f[t-n+1]^T, \ldots, f[t]^T)^T \\ e &= (v[t-n+1]^T, \ldots, v[t]^T, \varepsilon[t-n+1]^T, \ldots, \varepsilon[t]^T)^T, \end{aligned}$$

where $z \in \mathbb{R}^{n(l_y+l_u)}$, $x \in \mathbb{R}^{(n+1)l_x}$, $f \in \mathbb{R}^{nl_f}$ and $e \in \mathcal{N}(0, \Lambda_e)$ is an i.i.d. Gaussian vector with zero mean and $\Lambda_e \in \mathbb{R}^{n(l_e+l_v)\times n(l_e+l_v)}$ is a positive definite symmetric covariance matrix. Then a sliding window model of length n can be written as

$$Lz = Hx + Ff + Ne \tag{5}$$

where

$$L = \begin{pmatrix} 0 & 0 & \ldots & 0 & -B_u & 0 & \ldots & 0 \\ I & 0 & & 0 & -D_u & 0 & & 0 \\ 0 & 0 & & 0 & 0 & -B_u & & 0 \\ 0 & I & & 0 & 0 & -D_u & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & & 0 & 0 & 0 & & 0 & -B_u \\ 0 & \ldots & 0 & I & 0 & \ldots & 0 & -D_u \end{pmatrix},$$

$$H = \begin{pmatrix} A & -E & 0 & \ldots & 0 & 0 \\ C & 0 & 0 & & 0 & 0 \\ 0 & A & -E & & 0 & 0 \\ 0 & C & 0 & & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & & 0 & A & -E \\ 0 & 0 & \ldots & 0 & C & 0 \end{pmatrix}, F_n = \begin{pmatrix} B_f & 0 & \ldots & 0 \\ D_f & 0 & & 0 \\ 0 & B_f & & 0 \\ 0 & D_f & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & 0 & B_f \\ 0 & & 0 & D_f \end{pmatrix},$$

$$N = \begin{pmatrix} B_v & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & & 0 & D_\varepsilon & 0 & & 0 \\ 0 & B_v & & 0 & 0 & 0 & & 0 \\ 0 & 0 & & 0 & 0 & D_\varepsilon & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & B_v & 0 & 0 & & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & D_\varepsilon \end{pmatrix}.$$

The sliding window model (5) is a *static* representation of the dynamic behavior on the window given the time indexes $(k-n+1, ..., k)$.

To guarantee, given the sliding window model (5), that no noise-free residuals can be created, it is assumed that

$$(H \;\; N) \text{ has full row-rank.} \tag{6}$$

Assumption (6) is fulfilled, e.g., if all sensors have measurement noise. To simplify the computations, it is assumed that the covariance matrix $\bar{\Sigma}$ of variable $N_H Le$ is equal to the identity matrix, that is

$$\bar{\Sigma} = N_H N \Lambda N^T N_H^T = I \tag{7}$$

where the rows of N_H forms an orthonormal basis for the left null-space of matrix H. Note that any model satisfying (6) can be transformed into fulfilling $\bar{\Sigma} = I$. The choice of an invertible transformation matrix T is non-unique and one possibility is

$$T = \begin{pmatrix} \Gamma^{-1} N_H \\ T_2 \end{pmatrix} \tag{8}$$

where Γ is non-singular and satisfying

$$N_H N \Lambda N^T N_H^T = \Gamma \Gamma^T \tag{9}$$

and T_2 is any matrix ensuring invertibility of T.

It is convenient to eliminate the unknown variables x in (5) by multiplying with N_H from the left such that

$$N_H L z = N_H F f + N_H N e. \tag{10}$$

The model (10) is in an input-output form. For any solution z_0, f_0, e_0 to (10) there exists an x_0 such that it also is a solution to (5), and also if there exists a solution z_0, f_0, e_0, x_0 to (5) then z_0, f_0, e_0 is a solution to (10). Thus no information about the model behavior is lost when rewriting (5) as (10).

To quantify diagnosability performance, define the vector $r = N_H L z$. The vector $r \in \mathbb{R}^{n(l_y - l_x)}$ depends on the fault vector f and the noise vector e and represents the behavior of the model (5).

4.2 Quantified diagnosability performance

Let $p^i_{\theta_i}$ be the probability density function, pdf, describing the vector r when there is a fault f_i present in the system represented by the fault time profile θ_i.

The set of pdf's of r representing the fault mode f_i, corresponding to all possible fault time profiles θ_i is defined as

$$\mathcal{Z}_{f_i} = \left\{ p^i_{\theta_i} | p^i_{\theta_i} \text{ consistent with fault mode } f_i \right\}. \tag{11}$$

Each fault mode f_i is thus described by a set $\mathcal{Z}_{f_i}$ of all pdf's consistent with the fault mode. Consider two different sets, $\mathcal{Z}_{f_i}$ and $\mathcal{Z}_{f_j}$, for two fault modes f_i and f_j in Fig. 1. Assume that there is a measure to quantify the distance from a specific pdf $p^i_{\theta_i} \in \mathcal{Z}_{f_i}$ given a fault time profile θ_i to any $p^j_{\theta_j} \in \mathcal{Z}_{f_j}$. Then, the shortest distance from $p^i_{\theta_i}$ to any pdf in $\mathcal{Z}_{f_j}$ is a quantified isolability performance of a fault $f_i = \theta_i$ from the fault mode f_j.

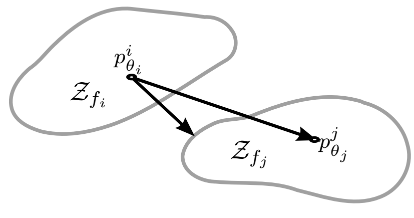

Fig. 1. A graphical visualization where the smallest difference between $p^i_{\theta_i} \in \mathcal{Z}_{f_i}$ and a pdf $p^j \in \mathcal{Z}_{f_j}$ is the quantified diagnosability measure.

To quantify the difference between the pdf's, $p^i_{\theta_i}$ and $p^j_{\theta_j}$, of r for two faults $f_i = \theta_i$ and $f_j = \theta_j$ the Kullback-Leibler divergence

$$K(p^i_{\theta_i} \| p^j_{\theta_j}) = \int_{-\infty}^{\infty} p^i_{\theta_i}(v) \log \frac{p^i_{\theta_i}(v)}{p^j_{\theta_j}(v)} dv = \mathrm{E}_{p^i_{\theta_i}} \left[\log \frac{p^i_{\theta_i}}{p^j_{\theta_j}} \right] \tag{12}$$

is used, see Kullback and Leibler (1951).

Then, to quantify isolability of a fault mode f_i with fault time profile θ_i from a fault mode f_j with an unknown fault time profile, a measure for isolability performance is defined as follows.

Definition 1. (Distinguishability). Given a sliding window model (5) of length n, under assumption (6), distinguishability $\mathcal{D}_{i,j}(\theta_i; n)$ of a fault f_i with a given fault time profile θ_i from a fault mode f_j is defined as

$$\mathcal{D}_{i,j}(\theta_i; n) = \min_{p^j \in \mathcal{Z}_{f_j}} K\left(p^i_{\theta_i} \| p^j\right). \quad (13)$$

Distinguishability can be used to analyze either isolability or detectability performance depending on whether $\mathcal{Z}_{f_j}$ describes a fault mode or the fault free case. Note that distinguishability is asymmetric in general, i.e., $\mathcal{D}_{i,j}(\theta_i; n) \neq \mathcal{D}_{j,i}(\theta_i; n)$, which is a natural property.

By using Theorem 2 and Theorem 3 in Eriksson et al. (2011a), distinguishability gives the maximum achievable FNR for any residual given the window model (5) of length n.

Theorem 2. For a window model (5) of length n under assumption (7), a tight upper bound for the fault to noise ratio of any residual based on (5) is given by

$$\mathcal{D}_{i,j}(\theta_i; n) \geq \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2$$

where $\lambda(\theta_i)/\sigma$ is the fault to noise ratio for a residual with respect to fault f_i and a fault time profile θ_i.

For a sliding window model (5) an explicit computation of (13) is stated in the following theorem.

Theorem 3. Distinguishability for a sliding window model (5) under assumption (6) is given by

$$\mathcal{D}_{i,j}(\theta_i; n) = \frac{1}{2}\|N_{\bar{H}} F_i \theta_i\|^2 \quad (14)$$

where $\bar{H} = (H \ F_j)$ and the rows of $N_{\bar{H}}$ is an orthonormal basis for the left null space of $\bar{H}$.

Proofs of Theorem 2 and Theorem 3 can be found in Eriksson et al. (2011b).

A detectability and isolability analysis of the descriptor model (3) can be made using distinguishability if the model is written as a sliding window model (5). The distinguishability measure depends on the window length n and the fault time profile θ_i.

5. THE SMALL EXAMPLE REVISITED

Consider again the example model (1). This time a minimal sensor set is sought which is a solution to the optimization problem (4). It is assumed that the faults to be detected are constants over time with amplitude one and a window model of length $n = 5$ is used when computing distinguishability.

Selecting the appropriate constraints, $\mathcal{D}^{\text{req}}_{i,j}(\bar{1}; 5)$ for each fault pair, $\{f_i, f_j\}$, can be difficult if $\mathcal{D}^{\text{req}}_{i,j}(\bar{1}; 5)$ contains many elements. A more convenient approach is to select $\mathcal{D}^{\text{req}}_{i,j}(\bar{1}; 5)$ as a fraction p of *maximum achievable distinguishability*, $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$ for each fault pair $\{f_i, f_j\}$, where $p \in [0, 1]$ is a scalar. In this way only one parameter is required for all elements. Note that there is still complete freedom in selecting $\mathcal{D}^{\text{req}}_{i,j}(\bar{1}; 5)$ for each fault pair individually. As when maximum fault isolability in Section 2.1 was determined, the maximum achievable distinguishability can be computed by including all sensors in $\mathcal{O}$. The computed $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$ is shown in Table 4. If $\mathcal{D}^{\text{req}}_{i,j}(\bar{1}; 5)$ is selected higher than $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$ for any fault pair, then the optimization problem can not be solved. By comparing Table 2 and Table 3 to Table 4 show that none of the two minimum sensors sets reaches $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$.

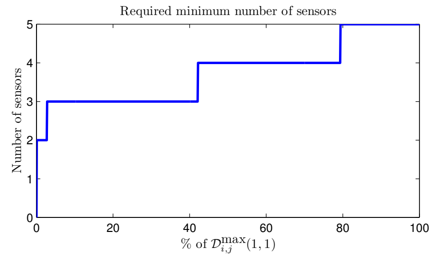

Fig. 2. The least number of sensors required to exceed a certain percentage of maximum distinguishability given the example model (1). Note that the number of sensors when required performance goes to zero is equal to the cardinality of the minimal sensor sets (2) from the structural analysis.

Table 4. Maximum achievable distinguishability for each fault pair, if the maximum number of sensors is used.

$\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$	NF	f_1	f_2	f_3	f_4
f_1	0.385	0	0	0.341	0.275
f_2	0.385	0	0	0.341	0.275
f_3	0.213	0.187	0.187	0	0.161
f_4	0.251	0.177	0.177	0.187	0

Assume that a minimal sensor set is to be found which achieves at least 50% of $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$ for each fault pair. That is, the sensor placement problem can be written as

$$\begin{aligned}&\min_{\mathcal{S} \subseteq \mathcal{O}} |\mathcal{S}| \\ &\text{s.t. } \mathcal{D}^{\mathcal{S}}_{i,j} \geq 0.5 \times \mathcal{D}^{\max}_{i,j}(\bar{1}; 5),\ i, j = 1, 2, 3, 4\end{aligned} \quad (15)$$

where $|\mathcal{S}|$ is the cardinality of $\mathcal{S}$, and $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$ can be found in Table 4. A global search gives that a solution to (15) is the unique optimal sensor set, of cardinality four, which measures the states: x_1, x_3, x_4, and x_5.

The analysis is expanded to see how the cardinality of the minimal sensor set, required to achieve a fraction p of $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$, depends on p. The result is presented in Fig. 2. Note that the minimum number of required sensors coincides with the cardinality of the minimal sensor sets (2) given by the deterministic analysis when $p \to 0+$.

Since there is only one minimal sensor set with four sensors achieving at least 50% of maximum distinguishability, the analysis in Fig. 2 gives that the minimal sensor set measuring the states: x_1, x_3, x_4, and x_5, achieves almost 80% of $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$. The number of sensors in the minimal sensor sets given by the structural analysis is two, which is not able to achieve more than 3% of $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$.

The result of the analysis in this section shows that the minimum cost sensor sets (2) given by the deterministic analysis results in a solution where the achieved diagnosability performance is relatively low given $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$ ($< 3\%$ of $\mathcal{D}^{\max}_{i,j}(\bar{1}; 5)$). By using minimum required distinguishability as the constraints of the optimization problem, a solution is found which better fits the requirements when designing a diagnosis system.

For the small system (1), a global search could be performed to find the solution. For larger systems, this is not realistic because of high computational complexity.

The number of sensor combinations, 2^k where k is the number of possible sensors, grows exponentially with the total number of sensors. A more efficient algorithm to reduce complexity is needed to find the optimal solution. An algorithm which iteratively adds new sensors to the solution, would be more appealing since it reduces the complexity. A heuristic is needed to implement such an iterative approach.

6. A GREEDY SEARCH APPROACH

A heuristic greedy search algorithm starts with an empty set and iteratively adds the sensor with the largest utility to the solution. The iteration continues until the solution fulfills the constraints. Thus, a utility function must be defined to use the greedy search heuristic.

In the iterative search, the heuristic adds the sensor s which best improves the previously selected set of sensors $\mathcal{S}$ to fulfill the constraints, i.e., the algorithm adds the sensor $s \in \mathcal{O} \setminus \mathcal{S}$ that maximizes the utility function $\mu(s)$. The utility function is here the sum, over all fault pairs, of the distinguishability improvements when adding a sensor s. There is no utility in improving distinguishability more than what is required by $\mathcal{D}_{i,j}^{\text{req}}$. Thus, the utility function can be written as

$$\mu(s) = \sum_{i,j} \max\left(\min\left(\mathcal{D}_{i,j}^{\text{req}}, \mathcal{D}_{i,j}^{\mathcal{S}\bigcup\{s\}}\right) - \mathcal{D}_{i,j}^{\mathcal{S}}, 0\right).$$

The algorithm SelMinSensSetGreedy for greedy selection of minimal sensor set is given below. The inputs to the algorithm are the model $\mathcal{M}$ in the form (3), a set of sensors $\mathcal{O}$ where each sensor measures one model variable and has a known noise variance, and a minimum required distinguishability $\mathcal{D}_{i,j}^{\text{req}}$. The output from the algorithm is a set of sensors $\mathcal{S}$. If the achieved distinguishability, $\mathcal{D}_{i,j}^{\mathcal{S}}$, given the set of sensors $\mathcal{S}$ fulfills the constraints $\mathcal{D}_{i,j}^{\text{req}}$ then the solution $\mathcal{S}$ is returned. If $\mathcal{D}_{i,j}^{\text{req}}$ is lower than the maximum achievable distinguishability, given $\mathcal{M}$ and $\mathcal{O}$ then SelMinSensSetGreedy will always return a set of sensors $\mathcal{S}$ fulfilling the constraints in (4).

function SelMinSensSetGreedy$(\mathcal{M}, \mathcal{O}, \mathcal{D}_{i,j}^{\text{req}}(\theta_i; n))$
  $\mathcal{S} := \emptyset$
  while $\mathcal{O} \neq \emptyset$ **do**
    $s^* := \arg\max_{s \in \mathcal{O}} \mu(s)$
    $\mathcal{S} := \mathcal{S} \bigcup \{s^*\}$
    $\mathcal{O} := \mathcal{O} \setminus s^*$
    if $\mathcal{D}_{i,j}^{\mathcal{S}}(\theta_i; n) \geq \mathcal{D}_{i,j}^{\text{req}}(\theta_i; n), \forall i, j$ **then**
      return $\mathcal{S}$
    end if
  end while
  return $\mathcal{S}$
end function

The complexity of the algorithm SelMinSensSetGreedy is linear in the number of sensors in $\mathcal{O}$. This approach is faster than a global search, however, the approach can of course not guarantee that the found solution is optimal.

7. SENSOR PLACEMENT USING GREEDY SEARCH

In this section, the greedy algorithm presented in Section 6 is applied to a slightly larger example, but still small enough to compare the solution to the global optimal solution.

Consider the sensor placement problem given a time-discrete linear model where *one* sensor can be selected to measure each unknown variable x_i. A minimum cost sensor set is to be found which achieves a minimum required distinguishability for each fault pair.

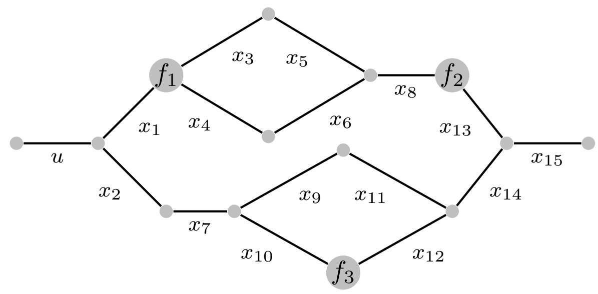

Fig. 3. A schematic overview of a model describing a static flow through a number of nodes. The input u is known and the flows through the branches x_i, $i = 1, 2, \dots, 15$, are unknown, and f_1, f_2, f_3 are three additive faults.

7.1 Model

The example is a static linear model describing a flow through a set of 15 branches visualized in Fig. 3. The input flow u is known and x_i represents the flow through each branch. For each node the sum of all branches is zero, e.g. $x_8 = x_5 + x_6$. The equations describing the flow through each node, has a model uncertainty which is assumed additive i.i.d. Gaussian $\mathcal{N}(0, 0.1)$, defines the model $\mathcal{M}$. The set of available sensors $\mathcal{O}$, one for each flow x_i, has measurement noise $\mathcal{N}(0, 1)$. There are three possible leaks added to the model, f_i, $i = 1, 2, 3$, and they are assumed to be additive in the equation describing the flow through the specific node, e.g., $x_3 + x_4 = x_1 + f_1$. To compute distinguishability, a window model of length one is assumed since the model is static and all fault time profiles are assumed amplitude one.

Note that, in the model described above, there are no equations in the model describing how the flow splits when the flow branches. This underdetermined model is analyzed first, and then in a second step the model is extended with equations describing how flow splits in the branches. This exactly determined model is analyzed to illustrate how diagnosability performance changes with modelling effort.

7.2 Analysis of the underdetermined model

Using the global search algorithm, the minimum number of sensors will depend on the minimum required distinguishability $\mathcal{D}_{i,j}^{\text{req}}(1;1)$. A higher $\mathcal{D}_{i,j}^{\text{req}}(1;1)$ requires more sensors added to the system to be fulfilled. The global search is used to analyze how the cardinality of $\mathcal{S}$ depends on $\mathcal{D}_{i,j}^{\text{req}}(1;1) = p\mathcal{D}_{i,j}^{\max}(1;1)$. The maximum achievable distinguishability for each fault pair is shown in Table 5. The solid line in Fig. 4 shows that the minimal number of sensors which achieves full deterministic isolability performance, i.e., four sensors, only is able to achieve approximately 30% of maximum distinguishability, for each fault pair. To achieve 70% of $\mathcal{D}_{i,j}^{\max}(1;1)$ for each fault pair a solution requires at least that 10 of 15 unknown variables, x_i, are measured by a sensor.

Table 5. Maximum achievable distinguishability for the underdetermined model in Fig. 3.

$\mathcal{D}_{i,j}^{\max}(1;1)$	NF	f_1	f_2	f_3
f_1	1.228	0	0.693	0.875
f_2	0.831	0.470	0	0.621
f_3	1.086	0.774	0.812	0

Assume that a minimal sensor set is to be found using the greedy algorithm which fulfills 50% of maximum distinguishability for each fault pair. The solution requires six

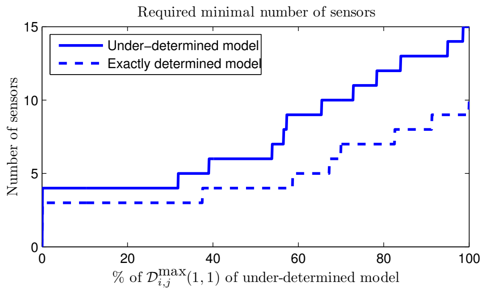

Fig. 4. The least number of sensors required to exceed a certain percentage of maximum distinguishability given the underdetermined model described in Figure 3, for the underdetermined case and the exactly determined case, given a global search.

sensors, selected by the greedy algorithm in the following order: x_{15}, x_{14}, x_8, x_7, x_{13}, and x_2. In this case the solution of the greedy algorithm is optimal since it has the same cardinality as the optimal solution given by the global search in Fig. 4.

The greedy algorithm always tries to find the sensor which best improves the distinguishability to fulfill the constraints and therefore some solutions could be missed. Consider, for example, in Fig. 3 that a set of sensors are to be found to isolate f_1 from f_2 and f_3. Assume that the optimal solution would be to measure x_1, x_3 and x_4, and that the algorithm has already selected x_1. Since neither only measuring x_3 or x_4 is enough to isolate f_1, a sensor measuring, for example, x_8 is selected. This sensor selection will improve the solution of the greedy search locally but will miss the global optimal solution. A solution to this could be to use the solution of a deterministic analysis and then improving the result by adding more sensors. Alternatively, more advanced search methods could be used, see e.g. Russell and Norvig (2003).

7.3 Analysis of the exactly determined model

Assume now that the flows in Fig. 3 where a branch is split are approximately equal, i.e., $x_3 \approx x_4$. This information is included in the underdetermined model, for example, by adding the equations $x_1 = x_2 + v_1$, $x_3 = x_4 + v_2$, and $x_9 = x_{10} + v_3$, where $v_i \sim \mathcal{N}(0,1)$, which makes the model exactly determined. The solution of the greedy algorithm, given the exactly determined model, which fulfills 50% of the maximum achievable distinguishability of the underdetermined model, in Table 5, is: x_{13}, x_{14}, x_8, and x_{15}. Note that the exactly determined model has a higher maximum achievable distinguishability than the underdetermined model, see Table 6. This is expected as additional process knowledge is incorporated in the model.

Table 6. Maximum achievable distinguishability for the exactly determined model in Fig. 3.

$\mathcal{D}^{\max}_{i,j}(1;1)$	NF	f_1	f_2	f_3
f_1	1.399	0	0.786	1.162
f_2	0.849	0.477	0	0.680
f_3	1.192	0.990	0.954	0

An extended analysis of the exactly determined case can be seen as a dashed line in Fig. 4. The number of sensors required to achieve a certain percentage of maximum distinguishability is lower compared to the underdetermined case. For the exactly determined case, only ten sensors are needed to achieve $\mathcal{D}^{\max}_{i,j}(1;1)$ of the underdetermined model, in Table 5, where 15 sensors are required for the underdetermined case. Comparing the results analyzing the underdetermined and the exactly determined model shows that better diagnosability performance can be achieved using fewer sensors at the price of more modeling work.

8. CONCLUSION

A key contribution in this paper, is the use of quantitative diagnosability analysis, distinguishability, to find optimal sensor sets for diagnosis. The sensor placement problem is formulated as a minimum cost optimization problem and a main observation is that the optimal solutions here differ significantly from solutions given by previously published deterministic methods.

The search space for the optimization problem is exponential in size and a heuristic greedy search algorithm is proposed as a solution to this for large problems. The algorithm iteratively adds the sensor which best improves diagnosability to fulfill the requirements.

Two examples are analysed to illustrate properties of the optimal solutions when using quantified diagnosability performance in the sensor placement optimization, e.g., how the number of sensors in the solution depends on the required diagnosability performance, and that better diagnosability performance can be achieved using fewer sensors by improving the model.

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